1923-1988

Ռաֆայել Ալեքսանդրյանը ծնվել է 1923թ. մարտի 29-ին Ալեքսանդրապոլում: 1945թ-ին, ավարտելով ԵՊՀ ֆիզմաթ ֆակուլտետը, գործուղվել է ՄՊՀ, որտեղ էլ սովորել է ասպիրանտուրայում Ս. Լ. Սոբոլևի ղեկավարությամբ: Թեկնածուական ատենախոսությունը պաշտպանել է 1949թ., իսկ դոկտորականը՝ 1962թ.:

Նրա նախաձեռնությամբ ԵՊՀ բացվեց դիֆերենցիալ հավասարումների ամբիոն, որն էլ նա գլխավորում էր երկար տարիներ: Տարբեր ժամանակներին Ռ. Ալեքսանդրյանը ղեկավարել է Մաթեմատիկայի ինստիտուտի դիֆերենցիալ հավասարումների բաժինը, եղել է ԳԱ հաշվողական կենտրոնի տնօրեն, ԵՊՀ պրոռեկտոր ու մեխմաթ ֆակուլտետի դեկան:

Նրա (Է. Միրզախանյանի հետ համատեղ) "Ընդհանուր տոպոլոգիա" դասագիրքը այսօր էլ լայն ճանաչում ունի:

Ռ. Ա. Ալեքսանդրյանը 1986թ. գիտական նվաճումների համար արժանացել է ԽՍՀՄ պետական մրցանակի (Ս. Լ. Սոբոլևի, Վ. Ն. Մասլեննիկովայի և Ս. Վ. Ուսպենսկու հետ համատեղ):

Ռ. Ալեքսանդրյանը վախճանվել է 1988թ, մարտի 22:

Մեջբերումներ հոդվածից, որը ստորագրել էին հայտնի Լ. Վ. Կանտորովիչը, Ս. Ն. Մերգելյանը, Օ. Ա. Օլեյնիկը, Ս. Լ. Սոբոլևը (УМН, 39(4), 1984).

Ռ.Ա. Ալեքսանդրյանը իր գիտական գործունեությունը սկսել է Ս. Լ. Սոբոլևի համակարգի համար բոլորովին նոր տիպի խառը խնդիրների լուծումների որակական հատկությունների հետազոտմամբ: Նա հանգեց Դիրիխլեի համասեռ խնդրի հետազոտմանը, որը դասական տեսանկյունից կոռեկտ չէ: Նրան հաջողվեց կառուցել ողորկ սեփական ֆունկցիաների բացահայտ ներկայացում և ապացուցել դրանց լրիվությունը:

Ռ. Ա. Ալեքսանդրյանի ուսումնասիրության էական փուլերից մեկը ին-տեգրոդիֆերենցիալ օպերատորի ընդհանրացած սեփական ֆունկցի-աների համակարգի կառուցման խնդրի հանգեցումն էր հատուկ դիֆեո-մորֆիզմների ընտանիքով ծնված դինամիկ համակարգերի էրգոդիկ հատկությունների ուսումնասիրմանը: Նրան հաջողվեց նշել յուրօրի-նակ ճանապարհ, որը հնարավորություն է տալիս կառուցել ինչպես ողորկ սեփական ֆունկցիաների, այնպես էլ սեփական ֆունկցիոնալ-ների որոշակի դասեր:

Ոչ էլիպտիկ դիֆերենցիալ օպերատորների փնջերի ուսումնասիրու-թյան ընթացքում պարզվեց, որ համապատասխան սեփական ֆունկ-ցիաները, որպես կանոն, խզվող են և անպայմանորեն պետք է դի-տարկվեն ընդհանրացված իմաստով: Ռ. Ա. Ալեքսանդրյանը առա-ջարկեց ոչ-էլիպտիկ եզրային խնդիրները ուսումնասիրել, գրգռելով քննարկվող հավասարումը փոքր, զուտ կեղծ գործակցով դեֆինիտ օպերատորով, ինչը շատ դեպքերում հնարավորություն է տալիս ու-սումնասիրել սկզբնական խնդիրը: Ռ. Ա. Ալեքսանդրյանը մշակեց հի-շատակված մոտեցումը ընդհանուր ինքնահամալուծ օպերատորների տեսության դեպքում և մտցրեց սպեկտրի կորիզի գաղափարը, ինչպես նաև նշեց սեփական ֆունկցիոնալների լրիվ համակարգի կառուցման համընդհանուր եղանակ:

Ռ. Ա. Ալեքսանդրյանի վերջին աշխատանքներում հիշատակված գա-ղափարները ստացան հետագա զարգացում արդեն կամայական ինք-նահամալուծ օպերատորների համար:

Ռ. Ա. Ալեքսանդրյանը իրականացրել է բազմակողմանի գիտա-կազ-մակերպչական գործունեություն՝ ուղղված Հայաստանում մասնակի ածանցյալներով դիֆերենցիալ հավասարումների, ֆունկցիոնալ անա-լիզի, և տոպոլոգիայի զարգացմանը: Այս առումով հատկապես մեծ դեր է կատարել Ռ.Ա. Ալեքսանդրյանի հիմնադրած ԵՊՀ դիֆերենցիալ հավասարումների ամբիոնը, ինչպես նաև ավելի քան 30 տարի գործող սեմինարը, որը նպաստել է հանրապետության երիտասարդ մաթե-մատիկոսների մեծ քանակի կայացմանը:

Ռ. Ա. Ալեքսանդրյանը հիանալի մանկավարժ էր՝ օժտված ամենաբարդ նյութը մատչելի և հրապուրիչ ներկայացնելու տաղանդով: Որտեղ և ինչ պաշտոնում էլ աշխատել է Ռ. Ա. Ալեքսանդրյանը, նրա պահանջկո-տությունը ինչպես իր, այնպես էլ ուրիշների նկատմամբ, լայն գիտական էրուդիցիան, ժամանակակից մաթեմատիկայի նորությունները ընկա-լելու մշտական ձգտումը զգալի չափով նպաստում էին Հայաստանի երիտասարդների գիտական հետազոտությունների մակարդակի բար-ձրացմանը և նրանց մտահորիզոնի ընդլայնմանը:

Rafael Alexandrian was born in 1923, on March 29. After graduating of Yerevan State University MathPhys faculty in 1945, he was sent to Moscow State University, where his advisor was S.L. Sobolev. He defended his Ph.D. thesis in 1949, and the doctor of Science thesis in 1962. On his initiative, in YSU was opened the chair of differential equations, which he led for many years.
During different periods, Alexandrian led the department of differential equations of the Institute of Mathematics, has been the director of the Computational Science Center, was Pro-rector and the Dean in YSU.
His (together with E. Mirzakhanyan) textbook "General Topology" is widely recognized till now.
In 1986 R. Alexandrian 1986 was awarded the USSR State Prize for Scientific Achievement (with S.L. Sobolev, V. N. Maslennikova, and S. V. Uspenski). R. Alexandrian died in 1988, on March 22.

Quotations from the article, which was signed by L. V. Kantorovich, S. N. Mergelyan, O. A. Oleynik, S. L. Sobolev (UMN, 39 (4), 1984):

He started his scientific activity with studying qualitative properties of the solutions of a completly new type of S. L Sobolev's mixed systems. He came to the study of a homogeneous Dirichlet problem, not well posed from the classical point of view. He succeeded to construct an explicit representation of the system of smooth eigenfunctions and to prove their completeness. One of the essential phases in the construction of the system operator was reducing the problem to the study of ergodic properties of a dynamic system generated by a special system of diffeomorphisms. He presented an original construction that enables to build both a class of smooth eigenfunctions and a class of eigenfunctionals.

In studying non-elliptic differential operator bundles it became clear that the spectral properties of corresponding expansions, as a rule, qualitatively depend of the domain type and the respective eigenfunctions should be interpret in the generalized sense. R. A. Alexandrian proposed an idea to study non-elliptic boundary-value problems with a small perturbation of the considered equation by a definite operator with small properly imaginary factor which in many cases enables to study the initial problem. R. A. Alexandrian treated this approach to the spectral theory of general self-adjoint operators, and particularly introduced the notion of the spectrum kernel, and showed the universal procedure of cunstracting the complete system of eigenfunctionals.

R. A. Alexandrian conducted a comprehensive scientific-organizational activity led to development in the republic the investigations on partial differential equations, functional analysis, topology. The chair of differential equations, established by R. A. Alexandrian and the regular workshop led by him during more than 30 years play a great role in formation of a whole number of young mathematicians. R. A. Alexandrian was an excellent teacher, endowed with the talent to present the most complex material in accessible and attractive way. Where and by whom was being worked R. A. Alexandrian, his demanding to himself and to others, his wider erudition, ongoing commitment to perceiving new ideas in the modern mathematics significantly contribute to the expansion horizons and mathematical culture, increasing the level of scientific research of young mathematicians of Armenia.

Կազմակերպչական կոմիտե՝

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Ա. Սահակյան, Վ. Սահակյան, Վ. Օհանյան

Ժողովածուն խմբգրված է Վ. Արզումանյանի կողմից

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Բովանդակություն/Content

Isotopic and isomorphic semirings

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Definition 1. *Semiring* is a set R equipped with two binary operations $+$ and $\cdot$ (called addition and multiplication), such that

1. $(R, +)$ is a commutative monoid with identity element 0:
   (a) $(a+b)+c = a+(b+c)$;
   (b) $0+a = a+0 = a$;
   (c) $a+b = b+a$.
2. $(R, \cdot)$ is a monoid with identity element 1:
   (a) $(ab)c = a(bc)$;
   (b) $1a = a1 = a$.
3. The left and right multiplications distribute over addition:
   (a) $a(b+c) = (ab)+(ac)$;
   (b) $(a+b)c = (ac)+(bc)$.
4. Multiplication by 0 annihilates R:
   $0a = a0 = 0$.

Definition 2. A semiring $(R, +, .)$ is called *commutative*, if $(R, .)$ is a commutative groupoid.

Examples.

- The motivating example of a semiring is the set $\mathbb{Z}_+$ of non-negative integers under usual addition and multiplication. This semiring is commutative.
- The set of all square matrices of given order with non-negative entries forms a (non-commutative) semiring under usual matrix addition and multiplication. More generally, one can consider the set of square matrices with entries from any other given semiring S, and the obtained semiring may be non-commutative, even if S is commutative.

- If A is a commutative monoid, the set $End(A)$ of endomorphisms of A forms a semiring, where addition is the pointwise addition and multiplication is the composition of functions. The zero and identity morphisms are respectively the neutral elements of the semiring.
- The set of ideals of a ring forms a semiring under addition and multiplication of ideals.
- Any bounded, distributive lattice is a commutative idempotent semiring under the join and meet operations.

Main result.
We consider the following concept of isotopy.
Rings (or semirings) $Q(+_1, \cdot_1)$ and $Q'(+_2, \cdot_2)$, are called isotopic if there exist bijective mappings

$$\alpha, \beta, \gamma : \; Q \to Q$$

such that

1. $\alpha(x \cdot y) = \beta(x) \circ \gamma(y)$
2. $\alpha, \beta, \gamma \in Aut[Q(+)]$

Theorem. *Isotopic semirings are isomorphic.*

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[2] Yu.M. Movsisyan, *Introduction to the theory of algebras with hyperidentities.* Yerevan State University Press, Yerevan, 1986.

Graph based orthogonal quasigroups in design of error correcting codes

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The main theoretical issue in communication and coding theory is to find the best rates of source or channel block codes that, assuming a known probabilistic model, guarantee arbitrarily small probability of error (or tolerable average distortion) when the block length is sufficiently large.
The type of a sequence $\mathbf{x} = (x_1, \ldots, x_N) \in X^N$ is a probability distribution $P = \{P(x) = (N(x \mid \mathbf{x}))/N, x \in X\}$, where $N(x \mid \mathbf{x})$ is the number of repetitions of symbol x among $(x_1, \ldots, x_N)$. Thus, the types define a partition of the set of all N -length sequences into classes according to their empirical distributions, [1].
Classes of equivalencies on graphs, generating a special class of permutations, so called cycles ([2]), stand for a special implementation of the method of types ([1]), in that they provide an alternative method to generate typical balanced sequences. In this sense, orthogonal quasigroups are mostly attractive, as orthogonal $(Q, *)$ and $(Q, \backslash)$ both are permutations on the same set Q.
As an alternative to the algebraical method of generating orthogonal quasigroups, the method of partitioning a simple connected graph into subsets with equal cardinalities is proposed. The sequence block size will dictate the order of the quasigroup, which, in own turn, will dictate the cardinality of the set of vertices of the graph. The orthogonal parastrophe in that case is the inverse permutation in appropriate cycles.
Given a quasigroup $(Q, *)$, balancing of the transmitted message is obtained by the quasigroup operation * on the given set. In order to detect/correct errors, we extend the input message $a_1\ a_2\ a_3 \ldots a_n$ to block $a_1\ a_2\ a_3 \ldots a_n\ d_1\ d_2\ d_3 \ldots d_n$, where $d_i = a_i * a_{i+1} \bmod n,\ i = 1, 2, \ldots, n$.
Error correction is made by exploiting the orthogonal parastrophe of the given quasigroup. As the elements of the quasigroup are closely correlated to each other, the latter provides diffusion of the error within the whole transmitted message, which makes it possible not only detect, but also correct several errors, providing construction of bounding minimum distance code so developed.
Let C be an $(n.k)$ code over $GF(q)$, with minimum distance d. We assume C is being used to correct t errors, where t is a fixed integer satisfying

$2t \leq d-1$. The decoder is implemented to be a bounded distance decoder, which looks for a codeword within distance t of the received word. Thus, t or fewer errors are recovered correctly, and with more than t errors, probability of decoding failures and errors, P_F and P_E, will be estimated. Estimation of P_F and P_E becomes difficult for $d-t$ errors, and consequently, with at least $d-t$ errors the received error pattern will be treated as random. It that case, the probability that a random error pattern will cause decoder error is given by

$$Q = \frac{(q^k - 1) \cdot V_n(t)}{q^n} = (q^{(-r)} - q^{(-n))}) \cdot V_n(t), \tag{1}$$

where $r = n - k$ is the code redundancy and

$$V_n(t) = \sum_{s=0}^{t} (\begin{matrix} n \\ s \end{matrix})(q-1)^n \tag{2}$$

is the volume of a Hamming sphere of radius t. This leads to the following estimation for P_E:

$$P_E \approx Q \cdot P_r\{\geq d - t \text{ errors}\}. \tag{3}$$

Probability that the error pattern has weight w leads to

$$P_E = \sum_{w=0}^{n} P_E(w) q_w, \quad P_F = \sum_{w=0}^{n} P_F(w) q_w, \tag{4}$$

where $P_E(w)$ and $P_F(w)$ denote the conditional probabilities of decoder error and failure respectively, given w channel errors. Notice that $P_E(w) = P_F(w) = 0$ for $w \leq t$ and $P_E(w) = 0$, $P_F(w) = 1$ for $t < w < d - t$. For $w \geq d - t$ we have $P_E(w) + P_F(w) = 1$, and so if $P_E(w)$ is known, $P_F(w)$ can be found, and vice versa.
A future work will be dedicated to derive upper bounds on the decoder error and failures probabilities for codes over a simple graph based on orthogonal quasigroups with provision of numerical results.

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[1] I. Csiszar and J. Korner, *Information Theory: Coding Theorem for Discrete Memoryless Systems.* Cambridge University Press, 2011.

[2] G. Margarov, Y. Alaverdyan. Partition and coloring in design of quasi-group equipped error-detecting codes. *Mathematical problems of Computer Science*, **38**: 74-75, 2012.

Invariant Imbedding in Integral Geometry and Tomography

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The well known in the theory of radiative transfer "Invariance principle" of V.A. Ambartsumian is often called "Invariant Imbedding", which term was proposed by famous US mathematician Richard Bellman. However Bellman entirely recognized V.A. Ambartsumian's priority. Thus, in the book "Invariant Imbedding and Radiative Transfer in Slabs of Finite Thickness" R.E. Bellman, R.E.Kalaba and M.C.Prestrud wrote that "due to V.A. Ambartsumian's pioneering work many of otherwise intractable problems were tamed, and great advances were made".
No less well known is V.A. Ambartsumian's contribution in the field which now is called Integral Geometry. In his paper [1] Nobel Laureate A.M. Cormack made the following comment pointing to the relation of V.A. Ambartsumian's work of 1936 to computer tomography: Ambartsumian gave the first numerical inversion of the Radon transform and it gives the lie to the often made statement that computed tomography would have been impossible without computers. In fact V.A. Ambartsumian was an independent discoverer of the Radon transform, a starting point of Integral Geometry. In the book "A Life in Astrophysics" published in New York in 1998 there is an Epilogue written by V.A. Ambartsumian that contains the following lines: "More recently, I learned that the invariance principle or invariant imbedding was applied in a purely mathematical field of integral geometry, where it gave birth to a novel, combinatorial branch. The resources of the invariance principle seem to be immense indeed!"
In Integral geometry that method was applied for the first time in the paper [2]. The same Invariant Imbedding derivation was reproduced in [3]. The solution of the Buffon-Sylvester Problem in R^3 ([4]) was also based on Invariant Imbedding. The monograph [5] contains several different proofs of that theorem, yet the first chapter there still points at Invariant Imbedding as the natural analytical approach that first led to a solution of a century-old Buffon-Sylvester Problem. So the tool of Invariant Imbedding led to discovery of the Combinatorial Integral Geometry tools.

At Vancouver International Congress of Mathematicians the invited talk [6] was on Stereology. The concepts of Mathematical Stereology and Mathematical Tomography are closely related, so it does not come as a surprise, that the solution to the Buffon-Sylvester Problem would one day find application in Mathematical Tomography. The ground-breaking paper [7] contains a result in loose terms described as follows.

The problem of reconstruction of general planar convex domain D on the basis of parallel X-rays (Hammer's X-ray problem) was posed in 1961 at the A.M.S. Symposium on Convexity: how many X-ray pictures of a convex body must be taken to permit its exact reconstruction? 45 years later Richard Gardner in his fundamental 2006 book "Geometrical Tomography" wrote that parallel X-rays in four different directions would do the job. Now [7] shows that in the usual asymptotical sense parallel X-rays in only three different directions can be enough for reconstruction of centrally symmetric convex domains. The accuracy of reconstruction increases as the directions of the three X-rays change, all three converging to some given direction. This approach is totally distinct from say, Radon transform inversion or approaches found in Gardner's book.

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On Cauchy problem for a system of ordinary differential equations, not reducible to normal system

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Introduction. In the report we consider a system of two linear differential equations with determinant not equal to zero identically, and not reducible to a normal system.

Section 1. The Cauchy problem for the following system of ordinary differential equations is studied

$$\begin{cases} L_1(p)u_1 + L_2(p)u_2 = f_1 \\ L_3(p)u_1 + L_4(p)u_2 = f_2 \end{cases}, \tag{1}$$

where $L_i(p)$ $(i = \overline{1,4})$ are polynomials of the differentiation operator $p = \frac{d}{dt}$ with constant coefficients, unknown functions u_1, u_2 and given functions f_1, f_2 being sufficiently many times differentiable. The system (1) can be represented in matrix form as

$$L(p)u = f, \text{where} \tag{2}$$

$$L(p) = \begin{bmatrix} L_1(p) & L_2(p) \\ L_3(p) & L_4(p) \end{bmatrix}, u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

Let $D(L)$ be the determinant of the matrix $L(\lambda)$ of the system (2).
In the paper [1] a necessary and sufficient condition for solvability of the system (1) is found when $D(L) \equiv 0$.
In the present report, the Cauchy problem for the system (1), which is not resoluble with respect to the highest derivative, and $D(L) \neq 0$ is studied.
Let us consider the homogeneous system

$$\begin{cases} L_1(p)u_1 + L_2(p)u_2 = 0 \\ L_3(p)u_1 + L_4(p)u_2 = 0. \end{cases} \tag{3}$$

Let λ be a zero of multiplicity k of the polynomial $D(L)$. According to the elimination method (see [2]), the solution $\overline{u} = [u_1, u_2]$ of the system, corresponding to λ, is represented by $\overline{u} = \overline{g}(t)e^{\lambda t}$, where $\overline{g}(t) = [g_1(t), g_2(t)]$, g_i $(i = 1, 2)$ are polynomials with orders not exceeding $k - 1$.

Section 2. Let n be the order of the polynomial $D(L)$, $\lambda_1, \cdots \lambda_r$ be the set of all its distinct zeros, and let m_i^j be the multiplicity of such λ_i, which are also the zeros of the polynomial L_j, $i = \overline{1,r}$, $j = \overline{1,4}$, $\omega_i^1 \equiv \min(m_i^1, m_i^3)$, $\omega_i^2 \equiv \min(m_i^2, m_i^4)$, $\omega_1 \equiv \sum_{i=1}^r \omega_i^1$, $\omega_2 \equiv \sum_{i=1}^r \omega_i^2$.

Theorem. *For any* $\alpha_i \in \mathbb{R}$, $i = \overline{0, \omega_1 - 1}$, $\beta_j \in \mathbb{R}$, $j = \overline{0, n - \omega_1 - 1}$, *the system* (1) *has the unique solution under any of the following initial conditions*

1. $\begin{cases} u_1^{(i)}(0) = \alpha_i & i = \overline{0, \omega_1 - 1} \\ u_2^{(j)}(0) = \beta_j & j = \overline{0, n - \omega_1 - 1} \end{cases}$, where $n > \omega_1 > 0$.

2. $\begin{cases} u_1^{(i)}(0) = \alpha_i & i = \overline{0, n - \omega_2 - 1} \\ u_2^{(j)}(0) = \beta_j & j = \overline{0, \omega_2 - 1} \end{cases}$, where $n > \omega_2 > 0$.

3. $u_1^{(i)}(0) = \alpha_i \quad i = \overline{0, \omega_1 - 1},$ where $n = \omega_1$.

4. $u_2^{(i)}(0) = \beta_i \quad i = \overline{0, n - 1},$ where $w_1 = 0$.

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Hyperspaces of compact convex bodies

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Introduction. A convex subset of a Euclidean space $\mathbb{R}^n$, $n \geq 1$, is called a *body* if it has a nonempty interior. In this paper we compute the hyperspace $cb(\mathbb{R}^n)$ of all compact convex bodies $A \subset \mathbb{R}^n$ and prove that the orbit space $cb(\mathbb{R}^n)/\operatorname{Aff}(n)$ under the natural action of the affine group $\operatorname{Aff}(n)$ is homeomorphic to the Banach-Mazur compactum $\operatorname{BM}(n)$.

Section 1. Let $cb(\mathbb{R}^n)$ be endowed with the Hausdorff metric:

$$d_H(A, B) = \max\left\{\sup_{b\in B} d(b, A),\ \sup_{a\in A} d(a, B)\right\}, \quad A, B \in cb(\mathbb{R}^n)$$

where d is the standard Euclidean metric on $\mathbb{R}^n$.
We will denote by $\mathbb{B}^n$ the closed n-dimensional Euclidean unit ball and by Q the Hilbert cube, i.e., $\mathbb{B}^n = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^n x_i^2 \leq 1\right\}$ and $Q = [0, 1]^\infty$.
It is easy to see that $cb(\mathbb{R}^1)$ is homeomorphic to the Euclidean plane $\mathbb{R}^2$. However, the topological structure of $cb(\mathbb{R}^n)$ for $n \geq 2$ has remained open until now.
In this report we provide a complete investigation of this problem. Here is our main result:

Theorem 1. *For any $n \geq 2$, the hyperspace $cb(\mathbb{R}^n)$ is homeomorphic to the product $Q \times \mathbb{R}^{n(n+3)/2}$.*

Our argument is based on some fundamental properties of the natural action of the affine group $\operatorname{Aff}(n)$ on $cb(\mathbb{R}^n)$ and also on the classical result in affine convex geometry about the John minimal-volume ellipsoid. Namely, it was proved by F. John [4] that for each $A \in cb(\mathbb{R}^n)$, there exists a unique minimal-volume ellipsoid $l(A)$ that contains A. It turns out that the map $l : cb(\mathbb{R}^n) \to E(n)$ is an $\operatorname{Aff}(n)$-equivariant retraction onto the subset $E(n)$ of $cb(\mathbb{R}^n)$ consisting of all n-dimensional ellipsoids. Denote by $L(n)$ the inverse image $L(n) = l^{-1}(\mathbb{B}^n)$. In other words, $L(n)$ is the subspace of $cb(\mathbb{R}^n)$ consisting of all convex bodies for which the unit ball $\mathbb{B}^n$ is the John minimal-volume ellipsoid.
As an intermediate result we first prove the following

Theorem 2. *$cb(\mathbb{R}^n)$ is homeomorphic to the product $L(n) \times E(n)$.*

Further, using profound results and methods of Infinite-Dimensional Topology, we prove that $L(n)$ is homeomorphic to the Hilbert cube Q, while $E(n)$ is homeomorphic to the Euclidean space $\mathbb{R}^{n(n+3)/2}$. Thus, we get that $cb(\mathbb{R}^n)$ is homeomorphic to the product $Q \times \mathbb{R}^{n(n+3)/2}$, as required.

Section 2. Another reason to study the hyperspace $cb(\mathbb{R}^n)$ is its close relationship with such classical objects as the Banach-Mazur compacta BM(n). Recall that BM(n) is the set of all isometry classes $[E]$ of all n-dimensional Banach spaces, topologized by the following metric best known in Functional Analysis as the Banach-Mazur distance:

$$d\big([E],[F]\big) = \ln\inf\big\{\|T\|\cdot\|T^{-1}\| \;\mid\; T : E \to F \text{ is a linear isomorphism}\big\}.$$

These spaces were introduced in 1932 by S. Banach [3] and they continue to be of interest. A. Pelczyński's old question of whether the Banach-Mazur compacta BM(n) are homeomorphic to the Hilbert cube (see [5, Problem 899]) was answered negatively for $n = 2$ by the first author [1]; the case $n \geq 3$ still remains open.
Here we prove that the orbit space $cb(\mathbb{R}^n)/\operatorname{Aff}(n)$ is homeomorphic to the Banach-Mazur compactum BM(n). The reader can find other results concerning the Banach-Mazur compacta and related spaces in [2].

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A solution of generalized cosine equation in Hilbert's fourth problem

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Introduction. The fourth problem in Hilbert's famous collection of 1900 asks particularly to give a description of axiomatically defined geometries, in which there exists a notion of length such that line segments are the shortest connections of their endpoints.
It was shown in [3] that it is the same to ask: *determine all complete, continuous and linearly additive metrics in* $\mathbf{R}^n$.
The modern approaches make it clear that the problem lies in the foundation of integral geometry, inverse problems and Finsler geometry.

We denote by $\mathbf{E}$ the space of planes in $\mathbf{R}^3$, $\mathbf{S}^2$ the unit sphere. We denote also by $[x]$ the bundle of planes containing the point $x \in \mathbf{R}^3$. Let μ be a locally finite signed measures on the space $\mathbf{E}$, which posses densities $h(e)$ with respect to the standard Euclidean motion invariant measure. To define the function h_x on $\mathbf{S}^2$ we consider the restriction of h onto $[x]$ as a function on the hemisphere. Then we extend the restriction to $\mathbf{S}^2$ by symmetry.
The following result (see [3]) belongs to A.V.Pogorelov.

Theorem. *If H is a smooth linearly additive Finsler metric in $\mathbf{R}^3$, then there exists a uniquely determined locally finite signed measure μ in the space $\mathbf{E}$, with continuous density function h, such that*

$$H(x,\Omega) = \int_{\mathbf{S}^2} |(\Omega,\xi)|\, h_x(\xi)\, d\xi \quad for \quad (x,\Omega) \in \mathbf{R}^3 \times \mathbf{S}^2. \tag{1}$$

Here h_x is the restriction of h onto $[x]$, $d\xi$ denotes the spherical Lebesgue measure on $\mathbf{S}^2$.

The measure μ is called a Crofton measure for the Finsler metric H. Thus for smooth linearly additive Finsler metrics, Pogorelov's result establishes the existence of a Crofton measure, generallly not positive. The equation (1) where H is a given even function and h is required, we call *generalized cosine equation.*
Note that in case when the measure μ is transition invariant on $\mathbf{E}$ (that

means $d\mu = m(xi) \cdot dp = h(\xi)d\xi \cdot dp$), the equation (1) represents the *cosine equation* (the zonoid equation).

The problem of finding a solution of (1) we reduce to the finding a solution of an other integral equation (so called *flag density equation*) appearing in Combinatorial Integral Geometry (see [4]).

In [1] (see also [2]) we propose an inversion formula for the solution of the integral equation using integral and stochastic geometry methods.

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Inverse crossed products

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We present a construction of inverse semigroups, associated with an endomorphism of a commutative group which allows to look at the different known constructions of crossed products from a unified point of view.

1. Preliminaries. We refer to ([1]) for the definition of positive defined functions (pdf) on a (unital) inverse semigroup S, which is a simple generalization of the notion for groups. The main properties of pdfs for groups with obvious changes could be transferred to the inverse semigroups.
The function δ, which takes the value 1 at the unit and 0 for the remaining elements is pdf. This function determines a positive form $< \cdot, \cdot >$ on the semigroup algebra $l^1(S)$ as $< s, t >= \delta(t^* s)$, $s, t \in S$. Then, applying the standard GNS-construction we obtain a representation referred as *A-regular*. It is quite different from the usual regular representation and acts in a much smaller Hilbert space. For groups the notions of regular and A-regular representations are coincide.
As a convincing example we consider the bicyclic semigroup $\mathfrak{B}$. It is the single generated (say, by the generator u) inverse semigroup with the relation $u^*u = e$, where u^* is a conjugate element (under the natural involution), e stands for identity element. Irreducible representations of this semigroup allow the complete description: there is a family of one-dimensional representations parameterized by the unit circle $\mathbf{S}^1$, and a single infinite dimensional coinciding from one hand with the A-regular representation (which is injective and irreducible in this case), and generating, from the other hand the Toeplitz algebra.

2. Inverse crossed product. Let G be an Abelian group, τ be an injective endomorphism of G. Denote by $\mathfrak{I}(G, \tau)$ the semigroup freely generated by G and $\mathfrak{B}$ with the following relations: $ug = \tau(g)u$ for any $g \in G$, and $u^*gu = \theta$ for $g \notin \tau(G)$, θ being zero element. It is easy to verify that the semigroup $\mathfrak{I}(G, \tau)$ is inverse with the involution imported from $\mathfrak{B}$ and G, where the conjugate element as usual is the group inverse. We call this semigroup the *inverse crossed product* of the group G and a semigroup $\mathbb{Z}_+$ (or $\mathfrak{B}$), associated with the action of the endomorphism τ. The structure of this semigroup could be described in details. We indicate only one property, which is related to the existence in $\mathfrak{I}(G, \tau)$ a canonical subsemigroup $\mathfrak{I}_0(G, \tau)$ with specified properties.

Proposition 1. *Inverse crossed product is $\mathbb{Z}$-graduated with respect to the subsemigroup $\mathfrak{I}_0(G,\tau)$.*

The following result allows to study this semigroup operator realized.

Proposition 2. *The A-regular representation of $\mathfrak{I}(G,\tau)$ is injective.*

3. Cartan subalgebras. Generally, the subalgebra $C^*(G)$ is not maximal commutative (Cartan) in $C^*(G,\tau)$, so it can be extended to a Cartan subalgebra adding some relations. Any representation of $C^*(G,\tau)$ preserving the relations is called *dynamical* (they could be extended to the quotient algebra with respect to the corresponding ideal).

We indicate such relations for two important cases:

- If $\tau_1(a) = a^2$ in the infinite single generated cyclic group G_1 (isomorphic to $\mathbb{Z}$) with the generator a, then $uu^* + a^{-1}uu^*a = e$.
- If $\tau_2(a_i) = a_{i+1}$ in the multiplicative group $G_2 = \sum \mathbb{Z}_2$ with second order generators $a_i, i = 1, 2, \ldots,$ then $uu^* + a_1uu^*a_1 = e$.

4. Examples.

- The dynamical system $(\mathbf{S}^1, \mathrm{mes}, T_1)$ with mes being the probability Lebesgue measure, and $T_1x = x^2$ is ergodic. The representation π_1 of $\mathfrak{I}(G_1,\tau_1)$ in $L^2(\mathbf{S}^1, \mathrm{mes})$, given by $(\pi_1(a)f)(x) = xf(x)$ (multiplication by the prime character), and $(\pi_1(u)f)(x) = f(T_1x)$ (Koopman operator, associated with T) is extended to the dynamical represenation.
- The system $([0;1], \mathrm{mes}, T_2)$ with Lebesgue measure, and $T_2x = 2x(\mathrm{mod} 1)$ is ergodic. The representation π_2 of $\mathfrak{I}(G_2,\tau_2)$ in $L^2(([0;1], \mathrm{mes})$, given by $(\pi_2(a_i)f)(x) = \varphi(x)f(x)$ (multiplication by the generators of the Walsh system), and $(\pi_2(u)f)(x) = f(T_2x)$ (Koopman operator, associated with T) is extended to the dynamical represenation.

Note that the regular representation of crossed products generated by both dynamical systems (see [2]) give the same wild factor $\mathrm{III}_{1/2}$. Obvious modifications of the examples lead to the $\mathrm{III}_{1/n}$ factors.

The considered algebraic systems are closely related to the Cuntz algebras $\mathcal{O}_n, n = 2, 3, \ldots,$ [3].

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On a question about the semigroup of endomorphisms of relatively free groups

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Let F be a relatively free group of some group variety and let End(F) and Aut(F) denote the semigroup of endomorphisms of F and the group of automorphisms of F respectively. Obviously, any automorphism of End(F) induces an automorphism of Aut(F) by restriction.
J. L. Dyer and E.Formanek proved that if F is a free group of finite rank $m > 1$, then the group Aut(F) is a complete group; that is, the center of Aut(F) is trivial and every automorphism of Aut(F) is inner. More recently, new proofs and various generalizations of this theorem have been obtained by D. G. Khramtsov, V. Tolstykh, M.R.Bridson and K. Vogtmann. Using the completeness of Aut(F), E.Formanek proved that every automorphism of End(F) is a conjugation by an element of Aut(F) (see [1]). Note that the question about the description of Aut(End(F)) for relatively free groups has been posed by B.Plotkin (see [1], [2]).
We have obtained the complete description of Aut(End(B(m,n))) for relatively free groups $B(m, n)$. By definition, the group $B(m, n)$ is the free group of rank m of the variety of all groups which satisfy the identity $x^n = 1$. The group $B(m, n)$ is a quotient group of the free group F_m on m generators by normal subgroup F_m^n generated by all n-th powers. Our main result is the following.

Theorem. If $S : End(B(m, n)) \to End(B(m, n))$ is an automorphism from $End(B(m, n))$, then there exists an $\alpha \in Aut(B(m, n))$ such that $S(\beta) = \alpha \circ \beta \circ \alpha^{-1}$ for all $\beta \in End(B(m, n))$, where $n \geq 1003$ is arbitrary odd period and $m > 1$.

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On a category of noncommutative quantum deformations of compact groups

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Preliminaries. Let $\mathcal{A}$ be a C^*-algebra. Denote by $\mathcal{A}\otimes\mathcal{A}$ the minimal C^*-tensor product of $\mathcal{A}$ on itself. A unital C^*-homomorphism $\Delta\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ is called a *comultiplication*, if the *coassociativity* relation holds:

$$(\Delta\otimes id)\Delta = (id\otimes\Delta)\Delta. \tag{1}$$

Then $(\mathcal{A},\Delta)$ is called a *compact quantum semigroup* [6, 1, 2].
Reduced semigroup C^*-algebra. Let G be a compact abelian torsion-free group, and Γ denote its dual discrete group. Take an arbitrary subsemigroup S in Γ, which generates group Γ, i.e.

$$\Gamma = \{a-b|\ a,b\in S\}.$$

Suppose that S contains the neutral element of the group Γ. Take a Hilbert space $l^2(S)$ with a standard orthonormal basis $\{e_a\}_{a\in S}$ [3]. Consider the regular representation of S on $l^2(S)$ by a map $a\to T_a$, where $T_a e_b = e_{a+b}$ for any $b\in S$. C^*-algebra, generated by the regular representation is denoted by $C^*_{red}(S)$ and called *a reduced semigroup C^*-algebra of S* [3]. Operators T_a and T_b^* are *trivial monomials* and generate an inverse semigroup of *monomials*.

Theorem 1. *Denote by K a commutator ideal in $C^*_{red}(S)$. The quotient algebra $C^*_{red}(S)/K$ is isomorphic to the algebra of continuous functions $C(G)$ on the group G.*

Quantum Semigroup. Define a map Δ on the semigroup of monomials in $C^*_{red}(S)$ by the relation

$$\Delta(\sum_{i=1}^{n}\lambda_i V_i) = \sum_{i=1}^{n}\lambda_i V_i\otimes V_i,$$

where V_i is a monomial.

Theorem 2. *Operator Δ extends to an embedding $\Delta\colon C^*_{red}(S)\to C^*_{red}(S)\otimes C^*_{red}(S)$. The pair $(C^*_{red}(S),\Delta)$ is a compact quantum semigroup.*

In terms of quantum semigroups, the result of Theorem 1 shows that G is a compact quantum subgroup in the compact quantum semigroup $(C^*_{red}(S), \Delta)$. Thus, $(C^*_{red}(S), \Delta)$ can be regarded as a noncommutative deformation of G, where "noncommutative" refers to the algebra of functions.

Category of quantum semigroups. Consider a category $\mathcal{S}_{ab}$ of discrete abelian cancellative semigroups, with arrows being semigroup morphisms. Denote by $\mathcal{QS}_{red}$ a category of compact quantum semigroups $(C^*_{red}(S), \Delta)$ for all $S \in Obj(\mathcal{S}_{ab})$. A morphism $\pi\colon (C^*_{red}(S_2), \Delta_2) \to (C^*_{red}(S_1), \Delta_1)$ is a unital *-homomorphism $\pi\colon C^*_{red}(S_1) \to C^*_{red}(S_2)$ which satisfies

$$\Delta_2 \pi = (\pi \otimes \pi)\Delta_1.$$

Theorem 3. *Category $\mathcal{QS}_{red}$ can be embedded in the category $\mathcal{S}_{ab}$.*

It is known that semigroup C^*-algebras of various semigroups, generating the same group Γ could be isomorphic [5, 4]. Generally, however, this isomorphism does not generate the isomorphism of the corresponding quantum semigroups. Thus, choosing different subsemigroups of a group Γ we obtain, by Theorem 3, the category of non-isomorphic quantum deformations of G.

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Defect Numbers of the Dirichlet Problem for Elliptic Equations

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Introduction. We consider the Dirichlet problem for higher order elliptic equations in the unit disk. The formulas for the defect numbers of the problem (numbers of linearly independent solutions of the homogeneous problem and linearly independent solvability conditions of the inhomogeneous problem) are obtained. For some classes of fourth order equations it is proved that these defect numbers may be equal only to zero or one. The connection with the Dirichlet problem for hyperbolic equations ([1], [2]) and for equations of general type ([3], [4]) is considered.

We consider in the unit disc $D = \{(x, y) : x^2 + y^2 < 1\}$ a higher order differential equation with constant coefficients:

$$\sum_{k=0}^{n} A_k \frac{\partial^n u}{\partial x^k \partial y^{n-k}}(x, y) = 0, \quad (x, y) \in D. \tag{1}$$

Here A_k are the complex constants ($A_0 \neq 0$). We suppose that all roots λ_j $(j = 1, \ldots, n)$ of the characteristic equation

$$\sum_{k=0}^{n} A_k \lambda^{n-k} = 0, \tag{2}$$

satisfy the condition $\lambda_j \neq \bar{\lambda}_j$ $(j = 1, \ldots, n)$, that is the equation (1) is elliptic. The solution of the equation (1) (in the class $C^n(D) \bigcap C^{(n-1,\alpha)}(\overline{D})$) satisfy on the boundary Γ the Dirichlet conditions

$$\left.\frac{\partial^k u}{\partial N^k}\right|_{\Gamma} = f_k(x, y), \quad (x, y) \in \Gamma, \quad k = 0, \ldots, n-1. \tag{3}$$

Here $f_k \in C^{(n-1-k,\alpha)}(\Gamma)$ are given functions, $\frac{\partial}{\partial N}$ is the derivative with respect to the inner normal to Γ. The problem (1), (3) for second order equations was considered in [5]-[7], where the fundamental difference between properly (when the numbers of the roots of (2) with positive and negative imaginary parts are equal) and improperly elliptic equations was found. It was proved ([7]) that the problem (1), (3) for the improperly elliptic equation (1) is not correct, and the class of functions this problem to be correct was found ([5]). Further, in [8] the problem was investigated for higher order properly elliptic equations. The explicit formulas for the defect numbers were found. Then in [9] the analogous results were obtained for some classes of fourth order improperly elliptic equations.

In the report another class of fourth order improperly elliptic equations is studied and the set of functions is defined for which the problem (1), (3) is correct. It is proved that the defect numbers in this class may only be zero (unique solvability) or one.

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Numerical solutions of the obstacle like problems

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Let consider an obstacle like problem which consists of minimizing some cost functional $\mathfrak{J}$ over the set of all admissible "deformations"

$$\mathbb{K} := \{v \in H^1(\Omega),\ \varphi \leq v, x \in \Omega, v(x) = g(x),\, x \in \partial\Omega\},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open subset with Lipschitz-regular boundary, φ is a some given obstacle.

In the presented report we propose an algorithm for solving the obstacle like problems based on a finite difference method.

О квазибегущих волнах

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Описание решений типа бегущей волны является одной из важных задач в моделях математической физики, в частности, в модели из теории пластической деформации о распространении бегущих волн для бесконечного стержня ([1]). Дискретный аналог такой системы моделируется поведением счетного числа шаров расположенных в целочисленных точках числовой прямой, соединенных между собой абсолютно упругой нитью. Такая система описывается конечно-разностным аналогом волнового уравнения с нелинейным потенциалом, моделирующего поведение бесконечного стержня под воздействием внешнего продольного силового поля

$$m_i\ddot{y}_i = \phi(y_i) + y_{i+1} - 2y_i + y_{i-1}, \qquad y_i \in \mathbb{R},\, m_i > 0,\, i \in \mathbb{Z},\, t \in \mathbb{R}, \quad (1)$$

где потенциал $\phi(.)$ задается гладкой функцией.

Определение. *Вектор-функция* $\{y_i(t)\}_{-\infty}^{+\infty}$, $t \in \mathbb{R}$ *с абсолютно непрерывными координатами называется решением системы* (1) *типа бегущей волны, если существует* $\tau > 0$ *, не зависящее от* t *и* i*, такое, что при всех* $i \in \mathbb{Z}$ *и* $t \in \mathbb{R}$ *выполнено равенство*

$$y_i(t+\tau) = y_{i+1}(t). \quad (2)$$

Константа τ *называется характеристикой бегущей волны.*

Для системы (1), в случае шаров с равными массами (однородный стержень) и гладкой периодической функцией $\phi(\cdot)$, были построены специальные классы решений типа бегущей волны. Методами теории возмущений было показано существование решения типа бегущей волны и для близких потенциалов. Обзор по работам такого направления для бесконечномерных систем с равными массами и потенциалами Френкеля – Конторовой, а также Ферми – Паста – Улама приведен в работе [2].
Вместе с тем, такой подход не позволяет описать пространство всех решений типа бегущей волны, а также их возможный рост.

В докладе представлен подход, при котором решения типа бегущей волны для системы (1) могут быть реализованы как решения однопараметрического семейства функционально-дифференциальных уравнений (ФДУ) точечного типа ([4]). При этом, систему (1) с потенциалом без особенностей удается исследовать при более общих предположениях на потенциал $\phi(\cdot)$ в виде условия Липшица. В рамках предложенного формализма удается изучить систему (1) не только в случае однородного стержня, но также и в случае неоднородного стержня.

Для однородного стержня ([3]) описание решений типа бегущей волны оказывается эквивалентным описанию всего пространства классических решений индуцированного однопараметрического семейства ФДУ точечного типа с параметром в виде характеристики бегущей волны. Удается описать решения типа бегущей волны, а также их возможный рост как по времени, так и по пространству. Стационарные решения исследуются на устойчивость.

Для неоднородного стержня ([4]), в силу тривиальности пространства решений типа бегущей волны, определяется их "правильное" расширение в форме решений типа "квазибегущей" волны. В отличие от однородного стержня, описание решений типа "квазибегущей" волны оказывается эквивалентным описанию уже всего пространства импульсных решений индуцированного однопараметрического семейства ФДУ точечного типа.

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О некоторых граничных свойствах нормальных функций

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В дальнейшем будем придерживаться общепринятых обозначений. Обозначим через D, Γ и $h(\xi,\varphi)$, соответственно, единичный круг $|z|<1$, единичную окружность $|z|=1$ и хорду единичного круга D, оканчивающуюся в точке $\varphi=e^{i\theta}\in\Gamma$ и образующую с радиусом в этой точке угол φ, $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$. Пусть $\triangle(\xi,\varphi_1,\varphi_2)$ обозначает подобласть круга D, ограниченную хордами $h(\xi,\varphi_1)$ и $h(\xi,\varphi_2)$. Если нас не интересует размер области $\triangle(\xi,\varphi_1,\varphi_2)$, мы будем обозначать его кратко $\triangle(\xi)$. Интерпретируя круг D, как модель плоскости в геометрии Лобачевского, обозначим через $\sigma(z_1,z_2)$ неевклидовое расстояние между точками z_1, z_2 из круга $D:\sigma(z_1,z_2)=\frac{1}{2}\ln\frac{1+u}{1-u}$, где $u=\left|\frac{z_1-z_2}{1-z_1\bar{z}_2}\right|$. Рассмотрим действительнозначную функцию $f(z)$. Для произвольного подмножества S круга D, для которого точка $\xi\in\Gamma$ является предельной точкой, обозначим через $C(f,\xi,S)$ предельное множество функции $f(z)$ в точке ξ относительно множества S, т.е. $C(f,\xi,S)=\cap\overline{f(S\cap U(\xi))}$, где пересечение берется по всем окрестностям $U(\xi)$ точки ξ, а черта означает замыкание множества относительно двухточечной компактификации $\bar{R}$ множества $R=(-\infty,+\infty)$ в виде отрезка, посредством добавления к точкам множества R символов $-\infty$ и $+\infty$. Скажем, что функция $f(z)$ имеет в точке $\xi\in\Gamma$ угловой предел α, если $C(f,\xi,\triangle(\xi))$ состоит из единственного значения α при любом $\triangle(\xi)$. Скажем, что действительнозначная функция $f(z)$, определенная в D, нормальна, если на группе T (состоящей из элементов $\{S(z):S(z)=e^{i\alpha}(z+a)\cdot(1+\bar{a}z)^{-1}\}$, где a – произвольная точка в D, α – произвольное действительное число) всех конформных автоморфизмов единичного круга D, порождаемое ею семейство функций $\Phi:\{f(S(z));S(z)\in T\}$, нормально в D в смысле Монтеля. Сформулируем основные результаты.

Теорема 1. *Пусть $f(z)$ – нормальная субгармоническая функция и для хорд $h(\xi,\varphi_1)$, $h(\xi,\varphi_2)$, существуют такие последовательности $\{z_n\}\in h(\xi,\varphi_1)$, $\{z'_n\}\in h(\xi,\varphi_2)$, для которых $z_n\to\xi$, $z'_n\to\xi$ при $n\to\infty$ и $\lim\limits_{n\to\infty}\sigma(z_n,z_{n+1})=\lim\limits_{n\to\infty}\sigma(z'_n,z'_{n+1})=0$. Если предельные множества $C(f,\xi,\{z_n\})$ и $C(f,\xi,\{z'_n\})$ ограничены сверху числом α,* *то предельное множество* $C\left(f,\xi,\overline{\triangle(\xi,\varphi_1,\varphi_2)}\right)$ *также ограничено сверху числом* α.

Следствие. *Пусть* $f(z)$ *– нормальная гармоническая функция и для некоторых хорд* $h(\xi,\varphi_1,$ $h(\xi,\varphi_2)$ *существуют такие последовательности* $\{z_n\}\in h(\xi,\varphi_1)$, $\{z'_n\}\in h(\xi,\varphi_2)$, *для которых* $z_n\to\xi$, $z'_n\to\xi$, *при* $n\to\infty$ *и* $\lim\limits_{n\to\infty}\sigma(z_n,z_{n+1})=\lim\limits_{n\to\infty}\sigma(z'_n,z'_{n+1})=0$. *Если предельные множества* $C(f,\xi,\{z_n\})$ *и* $C(f,\xi,\{z'_n\})$ *ограничены сверху (или снизу) числом* α, *то предельное множество* $C\left(f,\xi,\overline{\triangle(\xi,\varphi_1,\varphi_2)}\right)$ *также ограничено сверху (или снизу) числом* α.

Теорема 2. *Пусть* $f(z)$ *– нормальная гармоническая функция. Для того, чтобы функция* $f(z)$ *имела в точке* $\xi=e^{i\theta}$ *угловой предел* α, *необходимо и достаточно существование двух таких последовательностей* $\{z_n\}\in h(\xi,\varphi_1)$, $\{z'_n\}\in h(\xi,\varphi_2)$, $z_n\to\xi$, $z'_n\to\xi$ *при* $n\to\infty$, $\lim\limits_{n\to\infty}\sigma(z_n,z_{n+1})=\lim\limits_{n\to\infty}\sigma(z'_n,z'_{n+1})=0$, *для которых справедливо соотношение* $\lim\limits_{n\to\infty}f(z_n)=\lim\limits_{n\to\infty}f(z'_n)=\alpha$.

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Covariogram of a triangle and chord length distribution

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Let R^2 be the Euclidean plane and $D \subset R^2$ be a bounded convex body with inner points. The function $C(D, \cdot) : R^2 \to [0, \infty)$ defined by

$$C(D,h) = L_2(D \cap (D-h)), \quad h \in R^2,$$

is called the *covariogram* of D (where $L_2(\cdot)$ is the 2-dimensional Lebesgue measure in R^2), see [1]. G. Averkov and G. Bianchi showed that every planar convex body is determined within all planar convex bodies by its covariogram, up to translations and reflections, see [2].
The orientation-dependent chord length distribution is a function which depends on the lines parallel to $u \in S^1$ (S^1 is the circle of radius 1 centered at the origin). We denote by $b(D,u)$ the breadth of D in direction u (the distance between two parallel support lines in the direction). The random line which is parallel to u and intersects D has an intersection point with the line which is parallel to direction $u^{\perp}$ and passes through the origin. The intersection point is uniformly distributed in interval $[0, b(D,u)]$. Denote the orientation-dependent chord length distribution function of D in direction $u \in S^1$ by $F(D,u,t)$.
Let G be the space of all lines g in the Euclidian plane R^2, and let (φ, p) be the polar coordinates of the foot of the perpendicular to g from the origin. Let $\mu(\cdot)$ be the locally finite measure on G, invariant with respect to the group of all Euclidian motions (translations and rotations). It is well known (see [3]) that the element of the measure up to a constant multiple has the following form $\mu(dg) = dp\, d\varphi$, where dp is the one-dimensional Lebesgue measure and $d\varphi$ is the uniform measure on the unit circle. For each bounded convex domain D, we denote the set of lines that intersect D by $[D] = \{g \in G,\ g \cap D \neq \emptyset\}$ and we have (see [3]) $\mu([D]) = |\partial D|$, where $|\partial D|$ stands for the length of the boundary of D.
A random line in $[D]$ is one with distribution proportional to the restriction of μ to $[D]$. Therefore

$$F(D,t) = \frac{\mu(\{g \in [D],\ |g \cap D| \leq t\})}{|\partial D|}.$$

for any $t \in R^1$. The function $F_D(t)$ is called the *chord length distribution function* of D. We obtain the following results (see [4]):

1. The explicit form of the covariogram and orientation-dependent chord length distribution function for any triangle:

$$C(\Delta ABC, tu) = \begin{cases} S\left(1 - \frac{t}{t_{\max}(u)}\right)^2, & t \in [0, t_{\max}(u)], \\ 0, & t \geq t_{\max}(u) \end{cases}$$

$$F(\Delta ABC, u, t) = \begin{cases} 0, & t \leq 0, \\ \frac{t}{t_{\max}(u)}, & t \in [0, t_{\max}(u)], \\ 1, & t \geq t_{\max}(u), \end{cases}$$

   where S is the area of ΔABC, and $t_{\max}(u)$ is the length of the maximal chord in the direction $u \in S^1$.

2. The form of $F_D(t)$ for any triangle.

3. If an orientation-dependent chord length distribution function is known for a dense set of S^1, then we can uniquely recognize the triangle with respect to reflections and translations.

4. For any finite subset $A \subset S^1$, there are two non-congruent triangles with the same orientation-dependent chord length distribution function for any $u \in A$. It gives a negative answer to the problem from [5].

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On the Density of C_0^∞ in Some Multi - anisotropic Sobolev Spaces

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Let $A = \{\alpha^j = (\alpha_1^j, ..., \alpha_n^j)\}_1^M$ be a finite set of multi-indices. By the *Newton polyhedron* (*N.P.*) of a set A we mean the minimal convex hull (which is a polyhedron) $\Re = \Re(A)$, containing all points of A.
A polyhedron $\Re$ with vertices in N_0^n is *complete* ([1]) if $\Re$ has a vertex at the origin and one vertex in each of coordinate axes. The k-dimensional faces of $\Re$ are denoted by $\Re_i^k$, $(i = 1, \cdots, M_k;\ k = 0, 1, \cdots, n-1)$.
A face $\Re_i^k$ of $\Re$ is said to be *principal* if among the exterior (with respect to $\Re$) normals of this face there is one, which has at least one positive component. If among the exterior normals of the principal face $\Re_i^k$ there is one, whose all components are nonnegative (positive), then the face $\Re_i^k$ is said to be *regular (completely regular).* A polyhedron $\Re$ is said to be *regular (completely regular)* (see [2] or [3]), if $\Re$ is complete and all $(n-1)$-dimensional noncoordinate faces of $\Re$ are regular (completely regular).
Let $\Re$ be a complete polyhedron with vertices in N_0^n, $\Re^0$ be the set of its vertices, and let $p \in (0, \infty)$. Let $W_p^\Re = W_p^\Re(E^n)$ (respectively, $W_p^{\Re^0}$) denote the set of functions u, with the norms (see [4], or [5])

$$||u||_{W_p^\Re} = \sum_{\alpha \in \Re} ||D^\alpha u||_{L_p} \qquad (1),$$

and, respectively

$$||u||_{W_p^{\Re^0}} = \sum_{\alpha \in \Re^0} ||D^\alpha u||_{L_p}. \qquad (2)$$

The sets $W_p^\Re$ (respectively, $W_p^{\Re^0}$) with the suitable norms (1) and (2) we call *multi-anisotropic* Sobolev spaces. It turned out that for arbitrary collections A (polyhedron $\Re(A)$) the character of multi-anisotropic Sobolev spaces can be essentially different from the usual (classical) isotropic, or anisotropic Sobolev spaces. Namely, the following statements hold:

Theorem 1. Let $p \in (0, \infty)$, and $\Re = \Re(A)$ be a complete polyhedron of a set A. Then there exists a constant $C > 0$ such that for all $u \in C_0^\infty$

$$\sum_{\alpha \in \Re} ||D^\alpha u||_{L_p} \le C \sum_{\alpha \in \Re^0} ||D^\alpha u||_{L_p}. \qquad (3)$$

Theorem 2. Let N.P. $\Re = \Re(A)$ be regular. The set C_0^∞ is dense in $W_p^\Re$ if and only if the inequality (3) is valid for all $u \in W_p^\Re$.

Theorem 3. Let N.P. $\Re = \Re(A)$ be regular, then the inequality (3) is valid for all $u \in W_p^\Re$ and, consequently the set C_0^∞ is dense in $W_p^\Re$.

The following example shows that for nonregular polyhedron $\Re$ the multi-anisotropic Sobolev space $W_p^\Re$ in general is not semilocal (recall that a Banach space B is called semilocal if $u \in B$ and $\varphi \in C_0^\infty$ leades $\varphi\ u \in B$ (see, for instance [6]). Moreover, the set C_0^∞ is not dense in this space.

Example. Let $n = 2$ and $\Re$ is N.P. with the multi-indices (0,0), (1,0), (0,1), (2,1). It is easily seen that $\Re$ is a nonregular quadrangle.
Let $u(x) = u(x_1, x_2) = x_1^{4/3} + x_2$, and $\Delta_1 = \{-1 \leq x_1 \leq 1; -1 \leq x_2 \leq 1\}$. A simple computation shows that u, $D^{(1,0)}u$, $D^{(0,1)}u$, $D^{(2,1)}u$ belong to $L_2(\Delta_1)$, and $D^{(2,1)}u \notin L_2(\Delta_1)$.
Let $\psi \in C_0^\infty(\Delta_1), \psi(x) = \psi(x_1, x_2) = x_2$ for $x \in \Delta_{1/2}$.
Since $D^{(0,1)}\psi(x) = 1$ for $x \in \Delta_{1/2}$, we have $D^{(2,1)}(\psi(x)u(x)) \notin L_2(\Delta_1)$, i.e $\psi u \notin W_2^\Re(\Delta_1)$, which means that $W_2^\Re(\Delta_1)$ is not semilocal. Analogously, one can show that the set $C_0^\infty(\Delta_1)$ is not dense in $W_2^\Re(\Delta_1)$.

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On self-commutators of some operators

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Let A be a bounded linear operator acting in a Hilbert space $(\mathcal{H}, \langle \bullet, \bullet \rangle)$. The difference $AA^* - A^*A = C(A)$ is said to be the self-commutator of the operator A . This notion was investigated first probably by Halmos in [3]. It is proved in [1] that for any operator A the inequality

$$\|AA^* - A^*A\| \le \|A\|^2 \tag{1}$$

holds.

Example. Let S be the operator of simple unilateral shift. Then the self-commutator of the isometry S is the orthogonal projection on the first element of the basis, shifted by S, so $\|S^*S - SS^*\| = 1$ and $\|S\| = 1$, hence $\|S^*S - SS^*\| = \|S\|^2$, meaning that inequality (1), for some operators in fact is an equality.

As $C(A - \lambda I) = C(A)$ for any $\lambda \in \mathbb{C}$, the inequality (1) may be sharpened

$$\|AA^* - A^*A\| \le \inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|^2 . \tag{2}$$

Prasanna proved ([5]) that for any operator A

$$\inf_{\lambda \in \mathbb{C}} \|A - \lambda I\|^2 = \sup_{\|x\|=1} \left\{ \|Ax\|^2 - |\langle Ax, x \rangle|^2 \right\} . \tag{3}$$

According to [6], there exists a unique complex number c belonging to the closure of the numerical range $W(A)$ such that

$$\inf_{\lambda \in \mathbb{C}} \|A - \lambda I\| = \|A - cI\| .$$

It is proved in [2] that $c = \lim_{n \to \infty} \langle Ax_n, x_n \rangle$, where $\{x_n\}$ is a sequence of unit vectors, approximating the supremum in (3).

Let V be the Volterra integration operator in $L^2(0;1)$ defined by the formula

$$(Vf)(x) = \int_0^x f(t)\, dt.$$

Let us calculate the both sides of (2) for V.
Recall ([4], Problem 165) that $W(V)$ is bounded by the curve

$$t \mapsto \frac{1-\cos t}{t^2} \pm i\frac{t-\sin t}{t^2},\ 0 \le t \le 2\pi,$$

and $\|V\| = \frac{2}{\pi}$.
Let α be the least positive solution of the equation $\tan\alpha + \alpha = 0$. Then $c = 1/\alpha^2$, $\inf\limits_{\lambda \in \mathbb{C}} \|V - \lambda I\| = c^2 + c$. Calculations show that
$\alpha \approx 2.0287578381104 34$, $\lambda \approx 0.242962685095034$ and

$$\min_{\lambda} \|V - \lambda I\|^2 \approx 0.301993551443623.$$

The operator $C(V)$ is two dimensional self-adjoint operator with eigenfunctions $\left(3 \mp \sqrt{3}\right) x - 1$, corresponding to the eigenvalues $\left\{\frac{\sqrt{3}}{6}, -\frac{\sqrt{3}}{6}\right\}$, so $\|C(V)\| = \frac{\sqrt{3}}{6} \approx 0,288675134...$

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Noncommutative C^*-algebras and quantum semigroups, generated by the elementary inverse semigroup

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Preliminaries. We consider the set N^∞ of infinite sequences of natural numbers. Denote by $\overline{n} = \{n_i\}_{i=1}^\infty$ an arbitrary element in N^∞, $n_i \in \mathbb{N}$. Let $\{e_n\}_{n=1}^\infty$ be the canonical orthonormal basis in the Hilbert space $l^2(\mathbb{N})$. One can split the space $l^2(\mathbb{N})$ into a direct sum of Hilbert spaces $l^2(\mathbb{N}) = \bigoplus_{k=1}^{\infty} H_k$, such that $\dim H_k = n_k$, and $\operatorname{card}(\{e_n\}_{n=1}^\infty \cap H_k) = n_k$. Denote by $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ the right shift operator: $Te_n = e_{n+1}$ and define $T_{\overline{n}} = \bigoplus_{k=1}^{\infty} P_k T P_k$, where $P_k : l^2(\mathbb{N}) \to H_k$ is an orthogonal projection. Operator $T_{\overline{n}}$ is a partial isometry, and the semigroup generated by finite products of $T_{\overline{n}}$ and $T_{\overline{n}}^*$ is an inverse semigroup.
Denote by $\mathcal{T}_{\overline{n}}$ the C^*-subalgebra in $\bigoplus_{i=1}^{\infty} B(H_i)$, generated by operators $T_{\overline{n}}$ and $T_{\overline{n}}^*$. We associate every sequence $\overline{n}$ in N^∞ with the C^*-algebra $\mathcal{T}_{\overline{n}}$. The structure of these C^*-algebras is closely related to the structure of C^*-algebras introduced in [3, 4, 5].

Irreducible representations of $\mathcal{T}_{\overline{n}}$. Let us give the complete description of irreducible representations of the C^*-algebra $\mathcal{T}_{\overline{n}}$.

Lemma 1. *The C^*-algebra $\mathcal{T}_{\overline{n}}$ is unital for any $\overline{n} \in N^\infty$.*

The following is an analogue of the result in [7] which describes irreducible representations of the elementary inverse semigroup.

Theorem 1. *The irreducible representations of $\mathcal{T}_{\overline{n}}$ are the following:*

1. *Representations of dimensions $n_i \times n_i$, with $n_i \in \overline{n}$;*
2. *One infinite-dimensional representation;*
3. *A family of one-dimensional representations parameterized by elements of the unit circle.*

A category of C^*-algebras.
Define a category $\mathcal{T}_{N^\infty}$ of C^*-algebras $\mathcal{T}_{\overline{n}}$ for all $\overline{n} \in N^\infty$, with morphisms being unital *-homomorphisms of corresponding C^*-algebras. Denote by $\hom(\mathcal{T}_{\overline{n}}; \mathcal{T}_{\overline{m}})$ the set of all unital *-homomorphisms from $\mathcal{T}_{\overline{n}}$ to $\mathcal{T}_{\overline{m}}$.
Let us take the sequence $N = \mathbb{N} \setminus \{1\}$ in N^∞. The algebra $\mathcal{T}_N$ is an object in the category $\mathcal{T}_{N^\infty}$ such that $\hom(\mathcal{T}_N; \mathcal{T}_{\overline{m}})$ is a non-empty set for every sequence $\overline{m} \in N^\infty$.

Lemma 2. *The C^*-algebra $\mathcal{T}_N$ is a universally repelling object in the category $\mathcal{T}_{N^\infty}$.*

A unital C^*-homomorphism $\Delta\colon \mathcal{A} \to \mathcal{A} \otimes_{min} \mathcal{A}$ of a unital C^*-algebra $\mathcal{A}$ is called a *comultiplication*, if the *coassociativity* relation holds:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta. \tag{1}$$

Then $(\mathcal{A}, \Delta)$ is called a *compact quantum semigroup* [6].
In [1, 2] the Toeplitz algebra $\mathcal{T}$ is endowed with a compact quantum semigroup structure. It turns out, that the C^*-algebra $\mathcal{T}_N$ also can be regarded as a compact quantum semigroup.

Theorem 2. *There exists a *-homomorphism $\Delta : \mathcal{T}_N \to \mathcal{T}_N \otimes \mathcal{T}_N$ such that $(\mathcal{T}_N, \Delta)$ is a compact quantum semigroup.*

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C^*-algebras generated by inverse subsemigroups of the bicyclic semigroups

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Introduction. Coburn proved in 1967 that all C^*-algebras generated by non-unitary isometries are canonically isomorphic ([1]). This theorem was extended to the semigroups which are the positive cones of totally ordered Abelian groups by Murphy [2] and Douglas [3], whose works, in turn were recently generalized by M. Aukhadiev and V. Tepoyan [6] (see also [7, 8]). Coburn's result can be viewed as a theorem concerning non-unitary representations of the bicyclic semigroup.
In this report we describe the non-unitary representations of the inverse subsemigroups of the bicyclic semigroup.

Preliminaries. Let $l^2(\mathbb{Z}_+)$ be the Hilbert space of all complex-valued functions on $\mathbb{Z}_+$ satisfying the condition

$$\sum_{n=0}^{\infty} |f(n)|^2 < \infty.$$

The set of functions $\{e_n\}_{n=0}^{\infty}$, $e_n(m) = \delta_{n,m}$ forms an orthonormal basis in $l^2(\mathbb{Z}_+)$, where $\delta_{n,m}$ denotes the Kronecker symbol. Let T be the right shift operator on $l^2(\mathbb{Z}_+)$ and T^* be the left shift operator.
The semigroup S generated by products of operators T and T^* is the bicyclic semigroup. Since Fredholm indexes $\mathrm{ind_F}$ of the operators T and T^* are respectively equal to -1 and 1, we can define a homomorphism $\mathrm{ind_F} : S \to \mathbb{Z}$.

Let $S_n = \{U \in S : \mathrm{ind_F} U = n \cdot k, k \in \mathbb{Z}\}$ and $\mathcal{T}_n$ be the uniformly closed algebra of operators on $l^2(\mathbb{Z}_+)$ generated by all linear combinations of operators from S_n. Obviously, $\mathcal{T}_n$ is a C^*-subalgebra of the algebra $\mathcal{T}_1$, which is the Toeplitz algebra generated by T and T^*.
The semigroup S_n contains a subsemigroup $S_{n,0}$ generated by the operators T^n and T^{n*}. We denote by $\mathcal{T}_{n,0}$ the C^*-subalgebra of the C^*-algebra $\mathcal{T}_n$ generated by the semigroup $S_{n,0}$. One can easily check, that the algebra $\mathcal{T}_{n,0}$ is isomorphic to the Toeplitz algebra.

Main results.

Lemma 1. *Algebra $\mathcal{T}_n$ can be represented as a direct sum*

$$\mathcal{T}_n = \mathcal{T}_{n,0} \oplus \mathcal{T}_{n,1} \oplus ... \oplus \mathcal{T}_{n,n-1},$$

where each $\mathcal{T}_{n,j}$ is a C^*-algebra isomorphic to the algebra $\mathcal{T}_{n,0}$.

Lemma 2. *Let $\mathcal{K}_n$ be the C^*-algebra of compact operators in $\mathcal{T}_n$. Then, the following short exact sequence splits*

$$0 \to \mathcal{K}_n \to \mathcal{T}_n \to C(S^1) \to 0.$$

Theorem. The *C^*-algebra $\mathcal{T}_n$ has exactly n unitarily inequivalent, irreducible, infinite-dimensional representations and a family of one-dimen– sional representations, indexed by elements of the unit circle.*
These results can be used in quantum semigroups theory [4, 5].

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Infinite ergodic transformations

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Let T be a 1-1 measure preserving transformation defined on the σ-finite Lebesgue measure space $(X, \mathfrak{B}, m)$. We assume that T is *ergodic*, that is $TA = A \Longrightarrow m(A) = 0$, or $m(X \setminus A) = 0$.
When $(X, \mathfrak{B}, m)$ is a probability space, Birkhoff's ergodic theorem implies:

$$A, B \in \mathfrak{B} \Longrightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} m(T^i A \cap B) = m(A)m(B). \tag{1}$$

The transformation T is *strongly mixing* if

$$A, B \in \mathfrak{B} \Longrightarrow \lim_{n\to\infty} m(T^i A \cap B) = m(A)m(B). \tag{2}$$

Let us call a set A a *wandering* set if $T^i A \cap T^j A = \emptyset$ for $i \neq j$.
It is not difficult to show that, in general, if $(X, \mathfrak{B}, m)$ is a finite or an infinite measure space, then T satisfies the following:

- $m(A) > 0 \Longrightarrow$ for a.a. $x \in X$, $\exists$ infinitely many $n > 0$, such that $T^n x \in A$;
- T does not admit wandering sets of positive measure.

Let us call T an *infinite ergodic* transformation if $m(X) = \infty$. For such transformations Birkhoff's ergodic theorem implies:

$$m(A), m(B) < \infty \Longrightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} m(T^i A \cap B) = 0. \tag{3}$$

Let A be a set of positive measure.
The set A is called *weakly wandering* (w.w.) if for some sequence of integers $\{n_i\}$, $T^{n_i} A \cap T^{n_j} A = \emptyset$ for $i \neq j$.
In this case the sequence $\{n_i\}$ is called a w.w. sequence .
The set A is an *exhaustive weakly wandering* (e.w.w.) if $X = \bigcap_{i=0}^{\infty} T^{n_i} A(disj)$.
Similarly, the sequence $\{n_i\}$ is called an e.w.w sequence.

For infinite ergodic transformations using property (3) it can be shown:

- $\exists$ a sequence of integers $\{n_i\}$ such that
  $m(A) < \infty \Longrightarrow$ for a.a. $x \in X$, $T^{n_i}x \in A$ for finitely many i only;
- T accepts e.w.w. sequences.

For infinite ergodic transformations property (2) becomes:

$$m(A), m(B) < \infty \Longrightarrow \lim_{n\to\infty} m(T^i A \cap B) = 0. \tag{4}$$

Examples of infinite ergodic transformations that satisfy property (4) exist. The following two theorems show the significance of that property.

Theorem 1. *Let T be an infinite ergodic transformation that satisfies property (4). Then there exists an increasing sequence of integers $\{0 = N_0 < N_1 < N_2 < \cdots\}$ such that if $n_i - n_{i-1} \geq N_i$ for $i \geq 0$, then $\{n_i\}$ is a w.w. sequence.*

Theorem 2. *Let T be an infinite ergodic transformation that satisfies property (4). Then there exists an increasing sequence of integers $\{0 = N_0 < N_1 < N_2 < \cdots\}$ such that if $n_i - 2n_{i-1} \geq N_i$ for $i \geq 0$, then $\{n_i\}$ is an e.w.w. sequence.*

Theorem 2 motivates the following algebraic result in additive number theory. Let us say that a set $\mathbb{A} \subset \mathbb{Z}$ tiles $\mathbb{Z}$ if there exists a set $\mathbb{B} \subset \mathbb{Z}$ such that $\mathbb{A} \oplus \mathbb{B} = \mathbb{Z}$; that is, any $z \in \mathbb{Z}$ can be written as $z = a + b$ uniquely with $a \in \mathbb{A}$ and $b \in \mathbb{B}$.

Theorem 3. *Let $\mathbb{A} = \{n_i\}$ be an increasing sequence of integers satisfying $n_i - 2n_{i-1} \to \infty$. Then $\mathbb{A}$ tiles $\mathbb{Z}$.*

There exists an algebraic proof of Theorem 3.
The proofs of the above theorems are contained in the forthcoming book mentioned below where there exists a more detailed treatment of properties of infinite ergodic transformations.

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On a generalization of nonlinear pseudoparabolic variational inequalities

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Introduction. The report presents some generalization of the pseudo–parabolic variational inequalities. Results of existence, uniqueness and the regularity of the corresponding weak solution are presented.

Problem Statement. Let V be a Banach space and K be a nonempty closed, convex subset of V. A family of operators $A : D(A) \to V'$ is considered on a dense linear manifold $D(A) \subset V$. The problem is the following: for given $f \in V'$ find $u \in K \bigcap D(A)$ satisfying the inequality

$$(Au, v-u) \geq (f, v-u) \tag{1}$$

when the following conditions are hold

- Operator A may be represented in the form $A = L\Lambda + M$,
- $(-\Lambda) : D(A) \to V$ is a generator for linear semigroup $G(s) : V \to V$,
- $L : V \to V'$ is continuous and bounded,
- $(L\Lambda_h(\varphi - \psi), \varphi - \psi) \leq (L\Lambda_h\varphi - L\Lambda_h\psi, \varphi - \psi), \forall\varphi, \psi \in K$,
- $(L\Lambda_h\varphi, \varphi) \geq 0, \forall\varphi \in K$,
- $M : V \to V'$ is pseudomonotone operator in K, i.e. it is bounded in K and the continuous, $u_n \rightharpoonup u$ $(u_n \in K)$ and $\overline{\lim}(Mu_n, u_n - u) \leq 0$ imply $\underline{\lim}(Mu_n, u_n - v) \geq (Mu, u - v), \forall v \in V$,
- If the set K is non bounded, then operator M is coercive in K, i.e. $\exists v_0 \in K$ s.t. $\frac{(Mv, v-v_0)}{\|v\|} \underset{\|v\| \to \infty}{\to} \infty$.

Theorem 1. ***(Statement of weak problem)*** *If $u \in K \bigcap D(A)$ is a solution of the variational inequality (1), then it satisfies the following weak problem:*

$$\begin{cases} (L\Lambda v, v-u) + (Mu, v-u) \geq (f, v-u), \forall v \in K \bigcap D(\Lambda) \\ u \in K \end{cases} \tag{2}$$

Main Results.

Theorem 2. ***(Existence and uniqueness of weak solution)*** *Let all statements above be met, with v_0 in the condition of coercivity of the operator M belonging to $K \bigcap D(A)$, and let the condition of compatibility of Λ and K be also satisfied. Then there exists a solution for the weak problem (2). If the operator M is strictly monotone, then the weak solution is unique.*

Theorem 3. ***(Existence of strong solution)*** *Let all conditions of the theorem 2 be met, $int(K) \neq \emptyset$, and the weak solution $u \in D(A)$. Then u is a solution for the variational inequality (1) as well.*

Theorem 4. ***(Regularity of weak solution)*** *Let all conditions of the theorem 2 be met, $int(K) \neq \emptyset$, $V \subset V'$, and the following conditions be met:*

- $f \in D(\Lambda)$,
- $(Mu - Mv, u - v) \geq c \|u - v\|^2, \forall u, v \in V$,
- *The semigroup $G(s)$ can be extended to the space V',*
- $G(s)(Mv) = M(G(s)v), G(s)(Lv) = L(G(s)v), \forall v \in V, \forall s \geq 0$,
- $\exists \rho > 0, G(s)v + G^*(s)v - G^*(s)G(s)v + (\rho - 1)v \in \rho K, \forall v \in K, \forall s \geq 0$.

Then the weak solution of the variational inequality is a strong solution.

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Neyman-Pearson Principle for More than Two Hypotheses*)

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In the theory of statistical hypotheses the testing of each statement on the distribution of a studied random object is called a *statistical hypothesis*. The main problem is to determine which of given hypotheses is realized. A decision should be made by considering N independent identically distributed experiments called the samples, $\mathbf{x} = (x_1, x_2, ..., x_N)$ from $\mathcal{X}^N$. In the case of two simple hypotheses the best solution is given in the famous Neyman-Pearson Lemma (see[1]-[5]).

Since the many hypotheses case is actual in applications, we consider in the report the same principle for the case of three, or more hypotheses.

Theorem.(Generalization of Neyman-Pearson Lemma). *Let us consider the decision problem for three continuous distributions P_1, P_2, P_3 on a set $\mathcal{X}$. For given numbers $0 < \alpha_{1|1}, \alpha_{2|2} < 1$ we choose T_1 and T_2 and sets $\mathcal{A}_1, \mathcal{A}_2$ such that*

$$\mathcal{A}_1 = \{\mathbf{x} : \min(\frac{P_1(\mathbf{x})}{P_2(\mathbf{x})}, \frac{P_1(\mathbf{x})}{P_3(\mathbf{x})}) > T_1\} \textit{ and } 1 - P_1^N(\mathcal{A}_1) = \alpha_{1|1},$$

$$\mathcal{A}_2 = \overline{\mathcal{A}_1} \cap \{\mathbf{x} : \frac{P_2(\mathbf{x})}{P_3(\mathbf{x})} > T_2\} \textit{ and } 1 - P_2^N(\mathcal{A}_2) = \alpha_{2|2}.$$

Finally, we define

$$\mathcal{A}_3 = \mathcal{X}^N - (\mathcal{A}_1 \cup \mathcal{A}_2).$$

Consider error probabilities

$$\alpha_{l|m} = P_m^N(\mathcal{A}_l), \ l, m = \overline{1,3}, l \neq m, \textit{ and}$$

$$\alpha_{m|m} = P_m^N(\overline{\mathcal{A}_m}), \ m = \overline{1,3}.$$

———————————————————

*) *This text, at the request of the author, is given in the original version.*

In the matrix

$$\begin{pmatrix} \alpha_{1|1}, \alpha_{2|1}, \alpha_{3|1} \\ \alpha_{1|2}, \alpha_{2|2}, \alpha_{3|2} \\ \alpha_{1|3}, \alpha_{2|3}, \alpha_{3|3} \end{pmatrix}$$

there are six independent values, since it is obvious that

$$\alpha_{m|m} = \sum_{l \neq m} \alpha_{l|m}, m = \overline{1,3}.$$

The test determined by $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ is optimal in the sense that, for each other test determined by $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ the following relations hold

$$\mathcal{B}_l \bigcap \mathcal{B}_m = \emptyset, \quad \bigcup_{m=1}^{3} \mathcal{B}_m = \mathcal{X}^N$$

with the corresponding error probabilities $\beta_{l|m} = P_m^N(\mathcal{B}_l), \quad l, m = \overline{1,3}$, if $\beta_{1|1} \leq \alpha_{1|1}$ then $\max(\beta_{1|2}, \beta_{1|3}) \geq \max(\alpha_{1|2}, \alpha_{1|3})$, and if $\beta_{2|2} \leq \alpha_{2|2}$ then $\beta_{3|2} \geq \alpha_{3|2}$.

The proof of the Theorem is a development of the proof of the Neyman-Pearson lemma in the case of two hypotheses, see [2].

The Theorem can be extended to arbitrary number of hypotheses and to the case of discrete distributions of objects.

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Random Coding Bound for Secrecy E-capacity Region of the Multiple Access Channel With Confidential Messages

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Introduction. In the report, we study a two-user discrete multiple access channel with one confidential message (MACC). The system involves two sources, two encoders, one receiver. One user wishes to transmit the confidential messages while ensuring the eavesdropping user to be kept in total ignorance of it. The eavesdropping user also receives the channel output and hence may obtain the confidential information sent by the other user. Bounds for the secrecy capacity region of the MACC were found in [4]. Liang and Poor obtained the capacity bounds of the general multiple access channel with two confidential messages [2]. We determine an inner bound for secrecy E-capacity region of the MACC, under the requirement that the eavesdropping user is kept in total ignorance.

Preliminaries, Problem and Result. Messages $m_1 \in \mathcal{M}_{1,N}$, $m_2 \in \mathcal{M}_{2,N}$ should be transmitted to the receiver while ensuring the message m_2 to be kept secret from the user 1. The level of secrecy is measured by the equivocation rate at the eavesdropping user. The memoryless multiple access channel is characterized by products of the conditional probability distribution (PDs) for N using of the channel

$$W^N_{Y|X_1,X_2}(\mathbf{y}|\mathbf{x}_1,\mathbf{x}_2) \triangleq \prod_{n=1}^{N} W_{Y|X_1,X_2}(y|x_{1,n},x_{2,n}).$$

A *code* is a triple $(f_1,\, f_2,\, g)$, where f_1 is a deterministic encoder, f_2 is a stochastic encoder, $g : \mathcal{Y}^N \to \mathcal{M}_{1,N} \times \mathcal{M}_{2,N}$ is a deterministic decoder. The maximal probabilities of error of the code (f, g_1, g_2) are

$$e(f_1,\, f_2,\, g,\, W_{Y|X_1,X_2}) \triangleq \max_{m_1\in\mathcal{M}_{1,N},\, m_2\in\mathcal{M}_{2,N}} \Pr\big((g^{-1}(m_1,m_2))^c\big|m_1,\, m_2\big).$$

A code $(f_1,\, f_2,\, g)$ is characterized also by coding rates and equivocation rates (the functions log and exp are taken to the base 2)

$$R_i \triangleq \frac{1}{N}\log|\mathcal{M}_{i,N}|,\ i=1,\,2, \quad R_e \triangleq \frac{1}{N} H_{P_{1,2},W_{Y_1|X_1,X_2}}(M_{2,N}|Y_1,X_1),$$

where conditional entropy $H_{P_{1,2},W_{Y_1|X_1,X_2}}(M_{2,N}|Y_1,X_1)$ is the uncertainty of user 1 with respect to confidential message m_2.
A rates pair $(R_1,\ R_2)$ is called *E-achievable* for the MACC iff there exists codes sequence such that

$$\lim_{N\to\infty}\frac{1}{N}\log|\mathcal{M}_{i,N}| = R_i,\ i=1,\,2,$$

the reliability requirement

$$e(f_1,f_2,\,g,\,W_{Y|X_1,X_2}) \leq \exp\{-NE\},$$

and the secrecy constraints

$$\lim_{N\to\infty}\frac{1}{N}H_{P_{1,2},W_{Y|X_1,X_2}}(M_{2,N}|\mathbf{Y}_1,\mathbf{X}_1) \geq R_2,$$

are valid.
Secrecy E-capacity region $C_s(E)$ for maximal error probabilities is defined as the set of all E-achievable rates $(R_1,\ R_2)$.
Let P_{U_1} be a type of a vector $\mathbf{u}_1 \in \mathcal{U}_1^N$ and $P_{U_2|U_1}$, $P_{X_1|U_1}$, $P_{X_2|U_2}$ and $V_{Y|X_1,X_2}$ be conditional types. We define conditional PDs $P^1_{Y|U_2,U_1}$ and $P_{Y|U_2,U_1}$. Then we define the inner bound $\mathcal{R}_s^*(E)$ of secrecy E-capacity region $\mathcal{C}_s(E)$.

Theorem. *For all $E_1>0, E_2>0$, the region $\mathcal{R}_s^*(E)$ is an inner bound for secrecy E-capacity region of the MACC: $\mathcal{R}_s^*(E) \subseteq C_s(E)$.*

Corollary. *If $E=(E_1,\ E_2)\to 0$, the achievable region given in the Theorem coincides with the inner bound for the secrecy capacity region of the MACC of [4].*

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On the Shannon Cipher System with Noisy Channel to the Wiretapper Guessing Subject to Distortion Criterion

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Introduction. We examine the Shannon cipher system (SCS) with a noisy channel to the wiretapper. The wiretapper obtains the noisy version of the cryptogram and tries to guess encrypted plaintext with given accuracy. In each step of sequential guesses the wiretapper have testing mechanism. The security level of the encryption system is measured by the expected number of wiretapper's guesses. The upper and lower bounds are obtained for the guessing rate.

Section 1. The guessing problem for SCS was considered by Merhav and Arikan [1]. The same problem with distortion and reliability requirements was solved by Haroutunian and Ghazaryan [2]. We investigated the case of SCS with a noisy channel to the wiretapper [3].
In this paper we considere the combined model of the SCS studied in the papers [2] and [3]. The cryptographic system depicted in Fig. 1 is the SCS with a noisy channel to the wiretapper. The message after ciphering by the key-vector is transmitted to legitimate receiver via a public channel. The legitimate receiver can recover the original plaintext using the cryptogram and the key-vector which is communicated to decipherer by an extra secure channel. The wiretapper eavesdropping by the noisy channel gains a noisy version of cryptogram and tries to guess the plaintext on given degree of exactness without knowing the key. It is assumed that the wiretapper knows the source and channel distributions and encryption functions. The wiretapper tries to reconstruct source messages within given distortion measure and distortion level. For approximative reconstruction of secret information the wiretapper makes sequential guesses, each time applying a testing mechanism by which he can know whether the estimate is successful or not and stops it when the answer is affirmative. The security level of this system is measured by the expected number of the wiretapper's guesses needed before succeeding. We estimate the expectation of the number of guesses that the wiretapper make before succeeding.

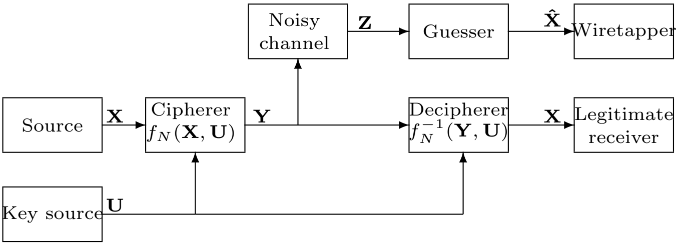

Fig. 1. The Shannon cipher system with noisy channel.

Our result is formulated in the following theorem. All definitions and notations can be found in the quoted paper [3].

Theorem. *For given PD P^*, conditional PDs W^*, V^*, and any key rate R_K, the following estimates are valid*

$$R(R_K, \Delta, P^*, W^*) \le \max_S \max_{P,Q,W} [\min\{R(P,\Delta), \lambda H_{Q,W}(Y|Z) + R_K\} \\ -D(P||P^*) - \lambda D(Q \circ W \| S \circ W^*)],$$

$$R(R_K, \Delta, P^*, W^*) \ge \max_P [\min\{R(P,\Delta), R_K\} - D(P||P^*)].$$

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Optimal Rank Test for Two-dimensional Data Homogeneity Analysis

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We propose a new method of construction of asymptotically optimal rank test for detection of changes in statistical properties of two-dimensional ordered sequence. This method allows to describe the class of positive quadrant dependent (PQD) bivariate distributions for which such associate measures as Spearman's and van der Waerden rank correlation coefficients and Kendall's concordance coefficient are optimal in sense of Pitman asymptotic relative efficiency [1].
When bivariate distribution function is represented by one parametric copula [4], then we can obtain in particularly, a new family of copulas which surfaces in unit cube lye between independence copula and the Farlie-Gumbel-Morgenstern (FGM) family of copulas for each fixed value of parameter.
Obtained results are applicable either for symmetric copulas or for threshold copulas investigated in [1] and [2].

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Chord length distribution function for convex polygons

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Let $\mathbb{G}$ be the space of all lines g in the Euclidean plane $\mathbb{R}^2$, (p, φ) be the polar coordinates of the foot of the perpendicular to g from the origin.
Let $\mu(\cdot)$ stands for a locally finite measure on $\mathbb{G}$ invariant with respect to the group of all Euclidean motions (translations and rotations). It is well known that an element of the measure up to a constant multiple has the following form $\mu(dg) = dp\, d\varphi$ (see [1]), where dp is the one dimensional Lebesgue measure, while $d\varphi$ is the uniform measure on the unit circle.
For each bounded convex domain $\mathbb{D}$ we denote

$$[\mathbb{D}] = \{g \in \mathbb{G} :\ \chi(g) = g \cap \mathbb{D} \neq \emptyset\}$$

and we have (see [1]): $\mu([\mathbb{D}]) = |\partial\mathbb{D}|$, where $|\partial\mathbb{D}|$ stands for the length of the boundary of $\mathbb{D}$.
Distribution function of the length of a random chord $\chi(g)$ is defined as

$$F(y) = \frac{1}{|\partial\mathbb{D}|}\,\mu(A_{\mathbb{D}}^{y}) = \frac{1}{|\partial\mathbb{D}|}\iint_{A_{\mathbb{D}}^{y}} d\varphi\, dp, \tag{1}$$

where $A_{\mathbb{D}}^{y} = \{g \in [\mathbb{D}] :\ |\chi(g)| \leq y\},\ \ y \in \mathbb{R}$.
Therefore, to obtain the chord length distribution functions for a bounded convex domain we have to calculate the integral in the right-hand side of (1). Explicit formulae for the chord length distribution functions are known only for the cases of disc, regular polygons (see [7]), rectangle (see [3]), and triangle (see [6]).
The result concerning the chord length distribution for general regular polygons was the first result for a class of domains rather than a concrete one. Thus, it has generalized all the cases known before, i.e. the cases of regular triangle (see [2]), square (see [3]), regular pentagon (see [4]) and regular hexagon (see [5]). To demonstrate its importance in applications a computer program is designed, which gives values of a chord length distribution function in the case of a regular n-gon for every natural number n, given the length of its side and the value of an argument. It is also shown that the chord length density function for a regular polygon with odd number of vertices has a jump only at the point 0, but when the

number of vertices is even the chord length density function also has a jump at the point equal to the distance between the parallel sides of a polygon (see [7]).

The result of a regular polygon incurred further generalization in [8], where an algorithm of calculating the chord length distribution function in the case of any convex polygon is obtained. The last formula is derived by using δ-formalism in Pleijel identity and the inclusion-exclusion principle. All results for the chord length distribution function that were known at the time of obtaining the result are particular cases of this one. The algorithm gives an opportunity to design similar computer program which will be able to calculate the chord length distribution function for any convex polygon when the coordinates of its vertices are given. In particular, due to the algorithm, an explicit expression for the chord length distribution function for a rhombus is obtained (see [8]).

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Some inverse spectral problems

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Let $L(q,\alpha,\beta)$ denote the Sturm-Liouville problem

$$-y'' + q(x)\, y = \mu y,\ x \in (0,\pi),\ q \in L^1_R[0,\pi],\ \mu \in C,$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0,\ \alpha \in (0,\pi],$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0,\ \beta \in [0,\pi),$$

and $\mu_n(q,\alpha,\beta)$, $n = 0,1,2,\ldots$ denote the eigenvalues of this problem. Let us note that arbitrary $\gamma \in (0,\infty)$ can be represented as $\gamma = \alpha + \pi k$, $\alpha \in (0,\pi]$, $k = 0,1,2,\ldots$, and arbitrary $\delta \in (-\infty,\pi)$ as $\delta = \beta - \pi m$, $\beta \in [0,\pi)$, $m = 0,1,2,\ldots$. The function μ_q defined on $(0,\infty) \times (-\infty,\pi)$ by the formula

$$\mu_q(\gamma,\delta) = \mu_q(\alpha + \pi k, \beta - \pi m) \stackrel{def}{=} \mu_{k+m}(q,\alpha,\beta)$$

we call the *eigenvalues function* (EVF) of the given family of operators $\{L(q,\alpha,\beta), \alpha \in (0,\pi), \beta \in [0,\pi)\}$.

We investigated the properties of this function. In particular, we found out that the uniqueness theorems (in inverse problems) of Ambarzumian, Borg, Marchenko, McLaughlin-Rundell and some others, we can consider as the properties of this real analytic function μ_q.

We also consider the constructive solution of inverse problem by EVF.
We introduce also the concept of eigenvalues function for Dirac canonical operators.
We have proved the uniqueness theorems in inverse problems for Dirac system. Their interpretations in terms of EVF are given.

Inverse spectral problem for a string equation with partial information on the density function

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We consider the inverse spectral problem for the string equation

$$-u'' = zp\,(x)\,u$$

given on the interval $(0;1)$, with self-adjoint boundary conditions and partial information concerning a positive density function $p \in W_1^2\,[0,1]$, i.e. $p'' \in L^1\,[0;1]$.

We investigate the situation where
1) the density p is known only on subinterval $[a;1]$ for some $a \in (0;1)$; and
2) only a part of one spectrum is given.

Under the suitable condition on the density of this part of spectrum, we have proved that the conditions 1) and 2) are sufficient the values of the density function p to be uniquely determined on the entire interval $[0;1]$.

A similar result was obtained in [1], however we consider more general boundary conditions.

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The solution of the inverse Sturm-Liouville problem for summable potential

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Let us denote by $\lambda_n^2(q,\alpha,\beta)$, $n=0,1,2,\ldots$ the eigenvalues of the Sturm-Liouville problem $L(q,\alpha,\beta)$:

$$-y''+q(x)\,y=\lambda^2 y,\ x\in(0,\pi),\ q\in L_R^1[0;\pi],\ \lambda\in C, \tag{1}$$

$$y(0)\cos\alpha+y'(0)\sin\alpha=0,\ \alpha\in(0;\pi),$$

$$y(\pi)\cos\beta+y'(\pi)\sin\beta=0,\ \beta\in(0;\pi).$$

By $\varphi(x,\lambda,\alpha)$ we denote the solution of (1), which satisfies the initial conditions

$$\varphi(0,\lambda,\alpha)=\sin\alpha,$$

$$\varphi'(0,\lambda,\alpha)=-\cos\alpha.$$

It is easy to see that

$$\varphi_n(x)\overset{def}{=}\varphi(x,\lambda_n,\alpha),\ n=0,1,2,\ldots$$

are the eigenfunctions of $L(q,\alpha,\beta)$.

The quantities

$$a_n(q,\alpha,\beta)=\int_0^\pi|\varphi_n(x)|^2dx,\ n=0,1,2,\ldots$$

are called the *norming constants.*

Theorem. *Let*

a) *a real sequence* λ_n *has the form* $\lambda_n=n+\frac{c}{n}+l_n$ *with* $c\in R$, $l_n=o\left(\frac{1}{n}\right)$, *and the function* $f(x)=\sum\limits_{n=1}^{\infty}l_n\sin nx$ *be absolutely continuous on the arbitrary subinterval* $[a;b]$ of $(0;2\pi)$,

b) *a positive sequence* a_n *has the form* $a_n=\frac{\pi}{2}+O\left(\frac{1}{n^2}\right)$, *as* $n\to\infty$.

Then there exist $q\in L_R^1[0;\pi]$, $\alpha\in(0;\pi)$, *and* $\beta\in(0;\pi)$ *such that* $\lambda_n^2=\lambda_n^2(q,\alpha,\beta)$, $a_n\sin^2\alpha=a_n(q,\alpha,\beta)$, $n=0,1,2,\ldots$.

This result is an extension of the corresponding theorems which have been proved in [1] - [4].

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The uniqueness of the solution of a Gelfand-Levitan equation

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Gelfand-Levitan (G.-L.) equation was constructed in [1] to solve the inverse Sturm-Liouville problem with the boundary conditions

$$y\left(0\right)\cos\alpha + y'\left(0\right)\sin\alpha = 0,\ \alpha \in \left(0,\pi\right),$$

$$y\left(\pi\right)\cos\beta + y'\left(\pi\right)\sin\beta = 0,\ \beta \in \left(0,\pi\right).$$

In [1] the authors proved the uniqueness of the solution of G.-L. equation using Levinson's theorem on the completeness of the system $\{\cos\lambda_n x\}_{n=0}^{\infty}$ in $L^2\left[0,\pi\right]$ under some conditions on $\{\lambda_n\}_{n=0}^{\infty}$.

We consider the inverse Sturm-Liouville problem with the conditions $\sin\alpha = 0$, $\sin\beta \neq 0$ or $\sin\alpha \neq 0$, $\sin\beta = 0$ and construct for these cases the corresponding G.-L. equation. Levinson's theorem is not sufficient to prove the uniqueness of the solution of G.-L. equation in these cases, however using the results of [2] we have managed to prove it.

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Some $\mathcal{L}$-convolution type integral operations on the semi-axis

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Let p be a continuous real function on $\mathbb{R}$, satisfying the condition

$$\int_{-\infty}^{\infty}(1+|x|)|p(x)|dx<\infty,$$

and $\mathcal{L}$ be a maximal Sturm-Liouville operator in $L_2(R)$ with potential p, generated by the differential operation

$$L\,y=-\frac{d^2}{dx^2}\,y(x)+p(x)\,y(x).$$

Let $\{V_k\}_{k=1}^{\nu}$ be an orthonormal system of eigenfunctions of the operator $\mathcal{L}$, and the functions $U^-(x,\lambda), U^+(x,\lambda)$ be eigenfunctions of the left and right scattering problem, respectively.
Denoting by $a_{ij}(\lambda)$, $(\lambda^2 a_{ij}(\lambda)\to 0\,\lambda\to\infty,\ i,j=1,2)$ continuous functions on $\mathbb{R}_+$, and by $\{c_k\}_{k=1}^{\nu}$ complex numbers, we define a kernel K by

$$\begin{aligned}K(x,t)=&\int_0^{\infty}(a_{11}(\lambda)\,U^-(x,\lambda)+a_{12}(\lambda)\,U^+(x,\lambda))\overline{U^-(x,\lambda)}d\lambda\ +\\&\int_0^{\infty}(a_{12}(\lambda)\,U^-(x,\lambda)+a_{22}(\lambda)\,U^+(x,\lambda))\overline{U^+(x,\lambda)}d\lambda\ +\\&\sum_{k=1}^{\nu}c_k\,V_k(x)\,\overline{V_k(t)},\ \ x,t>0.\end{aligned}\qquad(1)$$

Note that the kernel satisfies the following partial differential equation

$$\frac{\partial^2}{\partial x^2}K(x,t)-p(x)\,K(x,t)=\frac{\partial^2}{\partial t^2}K(x,t)-p(t)\,K(x,t)\qquad(2)$$

and corresponds to the particular case of the kernels of integral equations on the segment, introduced into consideration in [1]. The idea to study the corresponding equations on unbounded intervals (analogously with the difference kernel equations, called $\mathcal{L}$-convolution type) was partially realized in the papers [2-4].

In the report we study an integral operator $\mathcal{I}+\mathcal{K}$ acting in $L_2(\mathbb{R}_+)$

$$(\mathcal{I}+\mathcal{K})y = y(x) + \int_0^\infty K(x,t)y(t)\,dt,\; x>0 \tag{3}$$

with the kernels of the type (1).
The method of studying the problem is based on a operator identity connecting the operator (3) with a singular integral inverse shift operator.
We obtain the Fredholmness criterion of the operator $\mathcal{I}+\mathcal{K}$ and the formula for its index in terms of functions a_{ij}, the left and right reflection coefficient r^+, r^-, and the transmission coefficient t (see [5]).
In the simplest case $q=0$ we have

$$K(x,t) = h(x-t) + l(x+t),\; h \in L_1(R),\; l \in L_1(R_+).$$

In case of reflectionless potential p, under the condition $K(x,t-x) \to 0$, $x \to \infty$, $t \in R$, we present the solvability theory for the operator $\mathcal{I}+\mathcal{K}$.
As an example we consider the kernel corresponding to the reflectionless potential $p(x) = -2/\operatorname{ch}^2(x)$.

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Almost periodicity in the spectral analysis of representations generated by a generalized shift on a locally compact space

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Let S be a compact Hermitian operator acting on a weakly complete Banach space X. Then $\exp(itS)$ is an isometric representation of the group $\mathbb{R}$ on X. It was shown by Lyubich in [1,2] that a system of eigenvectors of S is complete if and only if for any $x \in X$ and $\varphi \in X^*$ the function $\varphi(\exp(itS)x)$ is Borh almost periodic on $\mathbb{R}$. Further, the extensions of this result for normal operators both in bounded and unbounded cases was obtained in [3] and [4].
Let Ω be a locally compact space, μ be a measure on Ω. In the report, using the methods of these papers we extend the mentioned results to the representations generated by generalized shift (g.s.) on Ω.
Let us consider a family of operators $\tau_\Omega = \{\tau_s, s \in \Omega\} \subset BL(L^1(\Omega, \mu))$ satisfying the specified properties as in [5,6].
Function f from the space $C_b(\Omega)$ of all bounded continuous functions is called *almost periodic with respect to g.s.* on Ω, if the families of functions $\{\tau_s f\}$ and $\{\tau_s^* f\}$ are compact in $C_b(\Omega)$. We denote by $C^\tau_{AP}(\Omega)$ the set of all continuous almost periodic functions with respect to τ_Ω which is a Banach subalgebra of $C_b(\Omega)$.

We present the following results.

Theorem 1. *Let X be a weakly complete Banach space, Ω be a locally compact space equipped with a measure μ, and T be an isometric representation of the space Ω in the space X generated by τ_Ω. A system of eigenvectors of T is complete in the space X if and only if for any $\varphi \in X^*$ and $x \in X$ the function $\varphi(T(\cdot)x) \in C^\tau_{AP}(\Omega)$.*

Theorem 2. *Let X be a reflexive Banach space, Ω be a locally compact space equipped with a measure μ, T be an isometric representation of the space Ω in X generated by τ_Ω. Assume that each weight subspace X_χ (χ being a character of the representation) is finite dimensional. If the system of eigenvectors of T is complete, then there exists a complete system of functionals which is biorthogonal to the union of the bases of all X_χ of T.*

As a nontrivial example of g.s. one can take a family of operators $\{\tau_s\}_{s\in\mathbb{R}}$ related to the differential operator $D, Du = u'' - \rho(x)u$, where ρ is an entire function, even on $\mathbb{R}$. For functions from $C^{(2)}(\mathbb{R}^2)$ such family can be defined as the solution $u(s,t) = \tau_t^s f(t)$ of the partial differential equation

$$\frac{\partial^2 u}{\partial s^2} - \rho(t)u = \frac{\partial^2 u}{\partial t^2} - \rho(s)u$$

with initial conditions

$$u(s,0) = f(s),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

A detailed survey of results concerning the problem can be found in [7].

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Convergence rate in the central limit theorem for martingale-difference random fields

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Let $(\xi_t) = \left\{\xi_t, t \in \mathbb{Z}^d\right\}$ be a martingale-difference random field ([1]), i.e. for any $t \in \mathbb{Z}^d$, $E\left|\xi_t\right| < \infty$ and $E\left(\xi_t/\sigma\left(\xi_s, s \in \mathbb{Z}^d \setminus \{t\}\right)\right) = 0$ a.s.
Let $\{V_n\}$ be a sequence of increasing d-dimensional cubes with side length n and let us put $S_{V_n} = \sum\limits_{t \in V_n} \xi_t$, $n = 1, 2, \ldots.$ The following central limit theorem for martingale-difference random fields is known (see [2]).

Theorem 1. *Let* (ξ_t) *be a homogeneous ergodic martingale-difference random field such that* $0 < \sigma^2 = E\xi_0^2 < \infty$. *Then*

$$\lim_{n\to\infty} P\left(\frac{S_{V_n}}{\sigma \cdot n^{d/2}} < x\right) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-u^2/2} du, \qquad x \in \mathbb{R}^1.$$

We present an estimate of convergence rate in the central limit theorem for martingale-difference random fields under some additional conditions.

Let a function $\varphi_I(\rho)$, $\rho \in \mathbb{R}^1$ be such that for any finite subset I of $\mathbb{Z}^d$, $\varphi_I(\rho) \to 0$ as $\rho \to \infty$. Let $\Im_S = \sigma(\xi_t, t \in S)$ be a σ-field generated by random variables ξ_t, $t \in S$, $S \subset \mathbb{Z}^d$. We say that a random field (ξ_t) satisfyes uniform strong mixing condition with coefficient $\varphi_I(\rho)$ if for any fixed finite $I \subset \mathbb{Z}^d$

$$\sup\left\{\left|P(A/B) - P(A)\right|, A \in \Im_I, B \in \Im_V, P(B) > 0\right\} \leq \varphi_I(\rho(I, V)),$$

where $\rho(I, V) = \inf\left\{|t - s|, t \in I, s \in V\right\}$, $|t - s| = \max\limits_{1 \leq i \leq d}\left|t^{(i)} - s^{(i)}\right|$.
The following theorem is the main result of our talk.

Theorem 2. *Let* (ξ_t) *be a homogeneous martingale-difference random field with finite phase space* X, *satisfying uniform strong mixing condition with coefficient* $\varphi_I(\rho)$, *such that* $\varphi_I(\rho) \leq |I|\, \varphi(\rho)$ *and* $\sum\limits_{j=1}^{\infty} j^{d-1}\varphi(j) < \infty$, *and let* $M\xi_0^2 = \sigma^2 > 0$*. Then*

$$\sup_x \left| P\left(\frac{S^{\xi}_{V_n}}{\sigma n^{d/2}} < x \right) - \Phi(x) \right| \le O\left(n^{-d/4}\right).$$

The next theorem, which is of independent interest, plays the basic role in the proof of Theorem 2.

Theorem 3. *Let* (ξ_t) *be a homogeneous martingale-difference random field with finite phase space* X*. Then for any* $k = 1, 2, ...$

$$M\left(S_{V_n}\right)^{2k-1} = O\left(|V_n|^{k-1}\right).$$

If, in addition, the random field (ξ_t) *satisfies the uniform strong mixing condition with coefficient* $\varphi_I(\rho)$*, such that* $\varphi_I(\rho) \le |I|\varphi(\rho)$ *and* $\sum\limits_{j=1}^{\infty} j^{d-1}\varphi(j) < \infty$*, then for any* $k = 1, 2, ...$

$$M\left(S_{V_n}\right)^{2k} = (2k-1)!!\left(M\xi_0^2\right)^k |V_n|^k \left(1 + O\left(|V_n|^{-1}\right)\right).$$

To prove the second statement of the Theorem 3 we use the following lemma.

Lemma 1. *Under conditions of Theorem 3, for any* $k = 1, 2, ...$ *and integer* $m_1, m_2, ..., m_k \ge 0$*, the following inequality holds*

$$\sum_{\substack{t_1, ..., t_k \in V_n \\ t_1 \ne t_2 \ne ... \ne t_k}} \left| M\xi_{t_1}^{m_1}\xi_{t_2}^{m_2}...\xi_{t_k}^{m_k} - M\xi_{t_1}^{m_1} M\xi_{t_2}^{m_2} \cdot ... \cdot M\xi_{t_k}^{m_k} \right|$$

$$\le C\,|V_n|^{k-1} \sum_{j=1}^{2n} j^{d-1}\varphi(j),$$

where $C > 0$ *is some constant.*

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On approximation of the backward stochastic differential equation

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Introduction. We consider the problem of approximation of the solution of the backward stochastic differential equation (BSDE) in the Markovian case [1]. Such equations are widely used in financial mathematics. We suppose that the trend coefficient of the diffusion process depends on some unknown parameter and the diffusion coeffcient of this equation is small. We propose an approximation of this solution based on the one-step MLE of the unknown parameter and we show that this approximation is asymptotically efficient in the asymptotics of "small noise".

Main result. We consider the following model. The observed diffusion process $X^T = (X_t, 0 \le t \le T)$ satisfies the equation

$$\mathrm{d}X_t = S(\vartheta, t, X_t)\,\mathrm{d}t + \varepsilon\sigma(t, X_t)\,\mathrm{d}W_t, \quad X_0 = x_0,\ 0 \le t \le T \qquad (1)$$

where $\vartheta \in \Theta = (\alpha, \beta)$. We are given two functions $f(t, x, y, z)$, $\Phi(x)$ and we have to find a couple of stochastic processes $(X_t, Z_t, 0 \le t \le T)$ such that the solution of the equation (*backward SDE*)

$$\mathrm{d}Y_t = -f(t, X_t, Y_t, Z_t)\,\mathrm{d}t + Z_t\,\mathrm{d}W_t, \qquad Y_0, \quad 0 \le t \le T \qquad (2)$$

at point $t = T$ satisfies the condition $Y_T = \Phi(X_T)$. Let us introduce a family of functions $\mathcal{U} = \{(u(t, x, \vartheta), t \in [0, T], x \in \mathcal{R}), \vartheta \in \Theta\}$ such that for all $\vartheta \in \Theta$ the function $u(t, x, \vartheta)$ satisfies the equation

$$\frac{\partial u}{\partial t} + S(\vartheta, t, x)\frac{\partial u}{\partial x} + \frac{\varepsilon^2\sigma(t, x)^2}{2}\frac{\partial^2 u}{\partial x^2} = -f\left(t, x, u, \sigma(x)\frac{\partial u}{\partial x}\right)$$

and condition $u(T, x, \vartheta) = \Phi(x)$. We propose the following solution. Fix some (small) $\delta > 0$ and introduce the *minimum distance estimator* $\vartheta^*_{\delta,\varepsilon}$ by the relation

$$\int_0^\delta \left[X_t - x_t\left(\vartheta^*_{\delta,\varepsilon}\right)\right]^2\,\mathrm{d}t = \inf_{\vartheta\in\Theta}\int_0^\delta \left[X_t - x_t(\vartheta)\right]^2\,\mathrm{d}t.$$

Here $x_t(\vartheta)$ is solution of (1) with $\varepsilon = 0$. This estimator is consistent and asymptotically normal (see Theorem 7.5 [2]).

Let us introduce the *one-step MLE*

$$\tilde\vartheta_{t,\varepsilon}=\vartheta^*_{\delta,\varepsilon}+\frac{\Delta_t\left(\vartheta^*_{\delta,\varepsilon},X_\delta^t\right)+\Delta_\delta\left(\vartheta^*_{\delta,\varepsilon},X^\delta\right)}{\mathrm{I}\left(\vartheta^*_{\delta,\varepsilon},x^t\left(\vartheta^*_{\delta,\varepsilon}\right)\right)},$$

where

$$\Delta_t\left(\vartheta,X_\delta^t\right)=\int_\delta^t\frac{\dot S\left(\vartheta,s,X_s\right)}{\sigma\left(s,X_s\right)^2}\left[{\rm d}X_s-S\left(\vartheta,s,X_s\right){\rm d}s\right],\quad t\in[\delta,T],$$

$$\Delta_\delta\left(\vartheta,X^\delta\right)=A\left(\vartheta,\delta,X_\delta\right)-\int_0^\delta A'_s\left(\vartheta,s,X_s\right){\rm d}s$$

$$-\frac{\varepsilon^2}{2}\int_0^\delta B'_x\left(\vartheta,s,X_s\right)\sigma\left(s,X_s\right)^2{\rm d}s-\int_0^\delta\frac{\dot S\left(\vartheta,s,X_s\right)S\left(\vartheta,s,X_s\right)}{\sigma\left(s,X_s\right)^2}{\rm d}s,$$

$$B\left(\vartheta,s,x\right)=\frac{\dot S\left(\vartheta,s,x\right)}{\sigma\left(s,x\right)^2},\qquad A\left(\vartheta,s,x\right)=\int_{x_0}^x B\left(\vartheta,s,z\right){\rm d}z,$$

$$\mathrm{I}\left(\vartheta,x^t\left(\vartheta\right)\right)=\int_0^t\frac{\dot S\left(\vartheta,s,x_s\left(\vartheta\right)\right)^2}{\sigma\left(s,x_s\left(\vartheta\right)\right)^2}{\rm d}s.$$

The approximation of the solution of BSDE is given in the following theorem.

Theorem. *Let the conditions of regularity be fulfilled then the processes* $\hat Y_t=u\left(t,X_t,\tilde\vartheta_{t,\varepsilon}\right)$, and $\hat Z_t=\varepsilon\sigma\left(t,X_t\right)u'_x\left(t,X_t,\tilde\vartheta_{t,\varepsilon}\right)$ *for the values* $t\in[\delta,T]$ *admit the representation*

$$\hat Y_t=Y_t+\varepsilon\dot u\left(t,X_t,\vartheta_0\right)\xi_t\left(\vartheta_0\right)+o\left(\varepsilon\right),$$
$$\hat Z_t=Z_t+\varepsilon^2\sigma\left(t,X_t\right)\dot u'_x\left(t,X_t,\vartheta_0\right)\xi_t\left(\vartheta_0\right)+o\left(\varepsilon^2\right),$$

where the Gaussian process

$$\xi_t\left(\vartheta_0\right)=\mathrm{I}\left(\vartheta_0,x^t\left(\vartheta_0\right)\right)^{-1}\int_0^t\frac{\dot S\left(\vartheta_0,s,x_s\left(\vartheta_0\right)\right)}{\sigma\left(s,x_s\left(\vartheta_0\right)\right)}{\rm d}W_s.$$

This estimators are asymptotically efficient (minimax) approximations of the solution (Y_t,Z_t) *of the BSDE* (2).

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C^*-subalgebras of Cuntz-Krieger algebras

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Introduction. In the last decades C^*-algebras generated by different families of partial isometries whose range and initial projections satisfy a specified conditions are in the focus of attention. Cuntz-Krieger algebra is one of the classical examples of such algebras. C^*-algebra generated by mapping also could be classed as ones. We are going to show that for every Cuntz-Krieger algebra there is a C^*-algebra generated by mapping, being the subalgebra of the first one.

Section 1. Cuntz in [1] considered the algebra $\mathfrak{O}_n$ generated by a family of noncommuting isometries $S_1, S_2, \ldots, S_n$ whose range projections are summed to the identity. The Cuntz-Krieger algebra construction (see [2, 3]) starts from a finite dimensional matrix $A = (A(i,j))$, $A(i,j) \in \{0,1\}$, such that every row and every column of A is non-zero. C^*-algebra $\mathfrak{O}_A$ is then generated by partial isometries $S_1, S_2, \ldots, S_n$ whose initial projections Q_i and range projections P_i satisfy the relations

$$P_iP_j = 0 (i \neq j), \quad Q_i = \sum_{j=1}^{n} A(i,j)P_j.$$

Cuntz algebra $\mathfrak{O}_n$ arises in this way from the matrix with all $A(i,j) = 1$.

A construction of C^*-algebra generated by mapping was proposed in [4, 5]. The algebra $C^*_\varphi(X)$ is generated by a mapping $\varphi : X \longrightarrow X$, where X is a countable set. The mapping φ is related to a family of partial isometries $\{U_k\}$ on $l^2(X)$. If cardinalities of preimages under the action of φ are bounded in common, then $C^*_\varphi(X)$ is a singly generated algebra with generator T_φ, $T_\varphi(f) = f \circ \varphi$ $(f \in l^2(X))$, which is a linear combination of these partial isometries. The last satisfy the relations:

$$\begin{cases} U_1^*U_1 + U_2^*U_2 + \cdots + = P_1 + P_2 + \cdots + P_n = P; \\ U_1U_1^* + U_2U_2^* + \cdots + = Q_1 + Q_2 + \cdots + Q_n = Q; \end{cases}$$

where P and Q are noncommuting projections defined by the mapping φ and $n = \sup_{y \in X} \operatorname{card} \varphi^{-1}(y)$.

Section 2. Let us fix a $n \times n$ matrix A with a property mentioned above. Following [2], we define Ω to be a set of all infinite sequences $\{\xi_i\}$ such that for every i, ξ_i does not exceed n and $A(\xi_i, \xi_{i+1}) = 1$. We consider $l^2(\Omega)$ with an orthonormal basis $\{e_{\overline{\xi}}\}_{\overline{\xi}\in\Omega}$, $\overline{\xi} = (\xi_1, \xi_2, \ldots)$. Partial isometries $\{S_i\}_{i=1}^n$ act on $l^2(\Omega)$ by the following way

$$S_k e_{\overline{\xi}} = \begin{cases} 0, \quad \text{if} \quad \xi_1 \neq k; \\ e_{\overline{\xi}'}, \overline{\xi}' = (\xi_2, \xi_3, \ldots) \quad \text{if} \quad \xi_1 = k; \end{cases}$$

and

$$S_k^* e_{\overline{\xi}} = \begin{cases} e_{\overline{\xi}'}, \overline{\xi}' = (k, \xi_1, \xi_2, \ldots) \quad \text{if} \quad (k, \xi_1, \xi_2, \ldots) \in \Omega; \\ 0, \quad \text{if} \quad (k, \xi_1, \xi_2, \ldots) \notin \Omega. \end{cases}$$

Proposition. *For every matrix A and family of partial isometries $\{S_i\}$ with a property mentioned in the section 1 there exists a mapping $\varphi : \Omega \longrightarrow \Omega$ such that $T_\varphi = S_1 + S_1 + \cdots + S_n$.*
We denote by $\mathfrak{C}^*(A)$ the C^*-algebra $C^*_\varphi(\Omega)$ generated by operator T_φ.

Theorem. *Let A be the matrix with all $A(i,j) = 1$. Then $\mathfrak{C}^*(A)$ is isomorphic to Toeplitz algebra.*

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General form of bounded linear functionals in some normed spaces

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In the report we describe the general form of bounded linear functionals in some functional spaces, which are introduced by the following way. Let $\mathfrak{M}_0$ be a class of functions on the interval $[0;1]$ which are one-sided continuous at the ends, and have one-sided finite limits $f(t-0)$, $f(t+0)$ at each point $t \in [0;1]$, so that $f(t) = 1/2(f(t-0) + f(t+0))$.
If at a point t there exist both one-sided derivatives f'_- and f'_+, we will put for (generalized derivative) $f'(t) = 1/2(f'_-(t) + f'_+(t))$. The derivatives (generalized) of a higher order are defined similarly.
Now we define inductively a decreasing family $\mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \mathfrak{M}_2, \ldots$ of spaces. A function f belongs to $\mathfrak{M}_{n+1}$, if $f \in \mathfrak{M}_n$, and there exists the generalized derivative $f' \in \mathfrak{M}_n$. Since the functions from all $\mathfrak{M}_n$ are bounded, the norm of $f \in \mathfrak{M}_n$ can be defined as $\|f\|_n = \sum_{i=1}^n \|f^{(i)}\|$, the last norm being the sup-norm.
The following result describes the conjugate space of the just introduced normed linear spaces $\mathfrak{M}_n$.

Theorem. *Each bounded linear functional L_n on the space $\mathfrak{M}_n$ has the form*

$$L_n(f) = \sum_{i=0}^{n} \int_0^1 f^{(i)}(t) dg_n(t),$$

for some bounded variation functions g_i, $i = 0, 1, \ldots, n$.

Boole-De Morgan Algebras

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An algebra $(Q; \{+, \cdot, ^{-}, ', 0, 1\})$ with two binary, two unary and two nullary operations is called a *Boole-De Morgan* algebra if $(Q; \{+, \cdot, ^{-}, 0, 1\})$ is a De Morgan algebra and $(Q; \{+, \cdot, ', 0, 1\})$ is a Boolean algebra and the two unary operations commute, i.e. $(\overline{x})' = \overline{(x')}$. This concept is introduced in [1, 2] under the name of Boolean bisemigroup.
Let us consider some natural examples of Boole-De Morgan algebras. First note that every Boolean algebra can be considered as a Boole-De Morgan algebra with two equal unary operations.
For a Boolean algebra $\mathfrak{B} = (Q; \{+, \cdot, ', 0, 1\})$ consider the direct product $\mathfrak{B} \times \mathfrak{B}$. Defining one more unary operation $^{-}$ on the set $Q \times Q$ by $\overline{(x, y)} = (y', x')$ we get the Boole-De Morgan algebra $\mathfrak{B} \times \mathfrak{B}$.
The set $\mathcal{T}(\mathfrak{A})$ of binary term operations of the nontrivial lattice $\mathfrak{A}$ is a Boole-De Morgan algebra (of order 4) with the operations defined below. For any binary terms $f(x, y)$ and $g(x, y)$ the binary operations are defined as the binary superpositions:

$$(f + g)(x, y) = f(x, g(x, y)),\ (f \cdot g)(x, y) = f(g(x, y), y).$$

The nullary operations are the terms y and x. The unary operations are the commutation and dualization. The commutation is defined by $\overline{f}(x, y) = f(y, x)$. To get the dual term $f'(x, y)$ of a binary term $f(x, y)$ we change all variables x by y and vice versa, and also change all operations $+$ by $\cdot$ and vice versa. So we obtain the Boole-De Morgan algebra $\mathcal{T}(\mathfrak{A})$. (For applications of binary superpositions see [3, 4, 5, 6].)
Let $\mathfrak{B} = (Q; \{+, \cdot, ', 0, 1\})$ be a Boolean algebra. Denote its dual Boolean algebra by $\mathfrak{B}^{\mathfrak{op}}$, i.e. $\mathfrak{B}^{\mathfrak{op}} = (Q; \{\cdot, +, ', 1, 0\})$. Consider the direct product $\mathfrak{B} \times \mathfrak{B}^{\mathfrak{op}} = (Q \times Q; \{\vee, \wedge, ', (0, 1), (1, 0)\})$ where $(x_1, y_1) \vee (x_2, y_2) = (x_1 + x_2, y_1 \cdot y_2)$, $(x_1, y_1) \wedge (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2)$, $(x_1, y_1)' = (x_1', y_1')$ for any $x_1, x_2, y_1, y_2 \in Q$. Defining one more unary operation $^{-}$ on the set $Q \times Q$ by $\overline{(x, y)} = (y, x)$ we get the Boole-De Morgan algebra $\mathfrak{B} \times \mathfrak{B}^{\mathfrak{op}}$.

Suppose $\mathfrak{A} = (Q; \{+, \cdot, ^{-}, ', 0, 1\})$ is a Boole-De Morgan algebra. From Stone's representation theorem for Boolean algebras it follows that there exists a set I such that the Boolean algebra $(Q; \{+, \cdot, ', 0, 1\})$ is isomorphic to a subalgebra of the Boolean algebra $(2^I; \{\cup, \cap, ', \emptyset, I\}) = \mathfrak{B}$ where for a set $X \subseteq I$ we define $X' = I \setminus X$. Let $\sigma : Q \to 2^I$ be an embedding of the mentioned Boolean algebra in $\mathfrak{B}$. We define an embedding of the Boole-De Morgan algebra $\mathfrak{A}$ in the Boole-De Morgan algebra $\mathfrak{B} \times \mathfrak{B}^{\mathfrak{op}}$ by the following rule:

$$\varphi(s) = (\sigma(s), \sigma(\overline{s})),\ s \in Q.$$

For a Boole-De Morgan algebra $(Q; \{+, \cdot, ^{-}, ', 0, 1\})$ we define one more unary operation * by the following way: $x^* = (\overline{x})' = \overline{(x')}$. It is easy to see that $(x+y)^* = x^* + y^*$, $(x \cdot y)^* = x^* \cdot y^*$, $\overline{x^*} = (\overline{x})^*$, $(x^*)' = (x')^*$. Thus the mapping $x \to x^*$ is an automorphism of the Boole-De Morgan algebra $(Q; \{+, \cdot, ^{-}, ', 0, 1\})$. Also it is easy to see that $x^* = x$ if and only if $x' = \overline{x}$.

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Theorems on convergence of sector sums of Haar and Walsh systems

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We use the following notations. For any two angles $0 \le \alpha, \beta \le \frac{\pi}{2}$ we put $G(\alpha, \beta) = \{(n, m) : \frac{m}{n} \in (\tan\alpha, \tan\beta)\}$, and let

$$S_{G(\alpha,\beta)}(x, y, W, f) = \sum_{(n,m)\in G(\alpha,\beta)} a_{nm} w_n(x) w_m(y)$$

$$S_{G(\alpha,\beta)}(x, y, \chi, f) = \sum_{(n,m)\in G(\alpha,\beta)} a_{nm} \chi_n(x) \chi_m(y),$$

where the systems $\{w_n,\, n = 1, 2, \ldots\}$ and $\{\chi_n,\, n = 1, 2, \ldots\}$ are respectively Walsh and Haar.

In the report we present the following results.

Theorem 1. *For any sequence of angles* $\alpha_k \searrow 0$ *there exists a function* $f \in L^2(0, 1)$ *such that* $S_{G(\alpha_k, \frac{\pi}{2})}(x, y, \chi, f)$ *diverges a.e. on* $[0, 1]$.

Theorem 2. *For any sequence of angles* $\alpha_k \searrow 0$ *there exists a function* $f \in L^2(0, 1)$ *such that* $S_{G(\alpha_k, \frac{\pi}{2})}(x, y, W, f)$ *diverges a.e. on* $[0, 1]$

Theorem 3. *For any sequence of angles* $\alpha_k \searrow 0$ *there exists a function* $f \in L^2(0, 1)^2$ *such that*

$$\sup_{1\le m\le n} \Big|\sum_{k=1}^{m} S_{G(\alpha_k, \frac{\pi}{2})}(x, y, W, f)\Big| \ge c \log_2 n.$$

Chebyshev Grids of Surfaces in Riemann decomposable spaces

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Let V_n be the n-dimensional Riemann space, equipped with the metric $ds^2 = g_{a_1b_1}(u^{c_1})du^{a_1}du^{b_1} + g_{a_2b_2}(u^{c_2})du^{a_2}du^{b_2}$, where $a_1, b_1, c_1 = \overline{1, n_1}$, and $a_2, b_2, c_2 = \overline{n_1 + 1, n}$. Obviously, the space V^n is the Cartesian product of two completely orthogonal positional manifolds V^{n_1} and V^{n_2}, where $n = n_1 + n_2$.

In the report we present the following results.

Theorem 1. *In order a coordinate grid of a decomposable space V_n be Chebyshev it is necessary and sufficient that the coordinate grids on the positional manifolds V^{n_1} and V^{n_2} be Chebyshev.*

Let X^m be a surface in V^{2m} with the projections X_1^m and X_2^m on basis manifolds. Assuming this we have.

Theorem 2. *Let the projections X_1^m and X_2^m are in affine correspondence. Then if the coordinate grid on one of them is Chebyshev, then the coordinate grids on the other two surfaces are also Chebyshev.*

Theorem 3. *Let the coordinate grids on any two of X^m, X_1^m and X_2^m surfaces are Chebyshev. Then the coordinate grid of the third surface will be Chebyshev as well.*

Theorem 4. *Let the surface X^m be completely geodesic in V^{2m} and let the coordinate grid on one of X^m, X_1^m and X_2^m surfaces is Chebyshev. Then the coordinate grid on the other two surfaces is also Chebyshev.*

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Permutation symmetry of the tensor categories in the theory of finite Fermi systems

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In quantum physics, along with the spatial symmetries, so called internal symmetries (gauge symmetries in abstract spaces) also play an important role. These symmetries are described by a compact group G of gauge transformations of the first kind, and are associated with an absolutely conserved quantities: generalized charges (electrical, baryonic, etc) in the case of an Abelian group, and the "isotopic charges" in non-Abelian case. Such a symmetry is identified by the dynamical superselection rules [1], prohibiting any transitions between different superselection subspaces (sectors) that are in one-to-one correspondence with the spectrum of G (the set of classes of irreducible unitary representations of G).

However, in physical systems which description is based on the theory of nets of abstract C^*-algebras of local observables (for example, in algebraic field theory), the study of superselection rules leads to the study of localized endomorphisms ρ, but not to representations of the group G. Such endomorphisms, due to localization, allow to select only those representations of the global algebra A of observables which satisfy a certain criteria of selection (the criteria of Doplicher-Haag-Roberts and Buchholz-Fredenkhagen). In this case, the endomorphisms ρ of A are generated by the finite dimensional Hilbert spaces H_ρ (generated by isometries ψ_i $(i = 1, 2, ..., d)$, namely, by the generators of Cuntz algebra).

The set of such endomorphisms forms a symmetric C^*-category of morphisms with intertwining operators as arrows [2]. We can define three algebraic operations in this category which are necessary ingredients of superselection structure: conjugation, permutation symmetry and composition of charges. This construction allows to build superselection theory on a strong mathematical basis, where the operation of passing to an antiparticle, the statistics of particles and the addition of charges give a clear physical interpretation to these mathematical operations. Moreover, the construction of the theory of superselection sectors in such way allows to involve into consideration the notion of crossed product of the algebra of observables with a compact group, and to construct the algebra of fields satisfying the Bose-Fermi statistics [3].

The main task of our work was to obtain specific representations of permutations θ and intertwining operators $T_1 \in (\rho_1, \rho_1')$, $T_2 \in (\rho_2, \rho_2')$ of the nontrivial endomorphisms ρ of the Cuntz algebra, and thereby to determine the permutation symmetry for localized endomorphisms of the fermionic (nucleon) system.
These endomorphisms are generated by linear transformations in the Fock space of the system $\varphi_i(f)$, $\varphi_i^*(f)$ $(i = 1, 2, ..., n)$, where f is a test function from the space of functions with compact support. These operators which generate the Fock representation of fermions are obtained with the help of representations of Cuntz algebra, using the so-called recursive technique proposed in [4].
It is also shown that the resulting unitary operators $\theta(H_\varrho, H_{\rho'})$ from $(H_\rho H_{\rho'}, H_{\rho'} H_\rho)$, and $T_1 \circ T_2 \in (\rho_1\rho_2, \rho_1'\rho_2')$, all satisfy the requirements for such operators (see e.g. the equations $(3.15 - 3.18)$ given in [2]).

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Pattern recognition by covariogram

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Is there a one-to-one correspondence between bounded convex bodies D and their chord length distribution functions $F_D(x)$? It was an old question of W. Blaschke whether the random chord length determine the convex body D uniquely up to a rigid motion (see [1]). This was disproved by Mallow and Clark (see [2]), who constructed two non-congruent bounded convex 12-gons with the same chord length distribution.
Another method consists of the consideration of chord length measurements not in the completely mixed form of the distribution but in preserving information about the line which generates the chord (see [3] and [6]). The applications in both geometric and computer tomography are well known (see [4]). For example, we can consider the case when the orientation and the length of the chords are observed. We refer to it as the orientation-dependent chord length distribution, that is for any fixed direction the distribution of the chord length is considered.
Let D be a convex body in n-dimensional Euclidean space R^n, that is a compact, convex subset of R^n, with nonempty interior. The n-dimensional Lebesgue measure in R^n is denoted by $L_n(\cdot)$. Various questions arise if one consider for each $u \in S^{(n-1)}$ (the unit sphere centered at the origin of R^n) the distribution of the length of the uniform random chord of the convex body D, with direction u. If $h \in R^n$, then $D + h$ denotes the translation of D by h. The covariogram of a convex body $D \subset R^n$ is the function $C(D, h) = L_n(D \cap (D + h))$.
In [5] G.Matheron conjectured that a planar convex body is uniquely determined (within the class of convex bodies) by its covariogram, up to translation and reflection.

G. Bianchi (see [7]) found counterexamples to the covariogram conjecture in dimensions greater than or equal to 4, and a positive answer for three-dimensional polytopes. The general three-dimensional case is still open.
Denote by $F_D(u, x)$ orientation-dependent chord length distribution function. Determination of convex body D by these distributions for all directions, is equivalent to the determination by its covariogram. G.Matheron (see [5]) obtained relationship between $F_D(u, x)$ and the covariogram. To obtain an explicit form of the covariogram for any convex body is very

difficult problem, but we can obtain covariograms for a subclass of convex bodies. The presence of these forms helps to solve many probabilistic problems, in particular calculate explicit forms of chord length distribution functions $F_D(x)$ and $F_D(u,x)$ (see, for instance, [8]). There exists a one-to-one corresponding between the set of all bounded convex bodies D and their $F_D(u,x)$ chord length distribution functions up to translations and the reflection.

The explicit form of covariogram for subclasses of convex bodies allows to obtain new properties of covariogram and generalizing these results we can determine what functions could be the covariograms of convex bodies. It is proved in [8] that for any finite A from S^1 there are two non-congruent domains for which orientation-dependent chord length distribution functions coincide for any direction from A (see also [9]). Moreover, in [8] explicit forms for covariogram and $F_D(u,x)$ for triangle are obtained. Finally, if we have the values of $F_D(u,x)$ for a dense set from S^1 then we can uniquely recognized the triangle (see [8]).

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On perfectness of the category of periodic groups

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For arbitrary variety Θ of algebras, it is denoted by Θ^0 the category of all free in Θ algebras $W = W(X)$ with finite set of generators X.
The study of automorphisms of the category Θ^0 is connected with the study of automorphisms of the semigroups $\mathrm{End}(W), W \in Ob(\Theta^0)$. The group of automorphisms $\mathrm{Aut}(W)$ consists of invertible elements of the semigroup End (W). There is an embedding $\mathrm{Aut}(W) \to \mathrm{Aut}(\mathrm{End}(W))$. The image of $\mathrm{Aut}(W)$ is the group of all inner automorphisms of the semigroup $\mathrm{End}(W)$.
The semigroup of endomorphisms $\mathrm{End}(G)$ of a group G consists of all homomorphisms

$$\mathrm{End}(G) = \{\varphi : G \mapsto G | \varphi \textit{ is a homomorphism}\}.$$

There is a map $\mathrm{Aut}(G) \mapsto \mathrm{Aut}(\mathrm{End}(G))$ given as follows:

$$\forall \varphi \in \mathrm{Aut}(G), \varphi \mapsto i_\varphi,$$

where

$$i_\varphi : \mathrm{End}(G) \mapsto \mathrm{End}(G)$$

and

$$i_\varphi(\alpha) = \varphi \circ \alpha \circ \varphi^{-1}.$$

Automorphisms of categories of free algebras for different varieties Θ and $W \in Ob(\Theta^0)$ were considered by various authors (see, f.e., [1]-[3]).

We study automorphisms of the category of free periodic groups.

Definition 1. *An automorphism $\varphi : C \mapsto C$ is called hereditary if for every $A \in Ob(C)$ the objects A and $\varphi(A)$ are isomorphic.*

Let φ be a substitution on objects of the category C such that A and $\varphi(A)$ are isomorphic for every $A \in Ob(C)$. Let us consider a function s that chooses an isomorphism $s(A) : A \mapsto \varphi(A)$ for any object A.
Define an automorphism $\hat{s} : C \mapsto C$ by the rule:

1. $\hat{s}(A) = \varphi(A)$, for every object A.
2. For every morphism $\nu : A \mapsto B$, $\hat{s}(\nu) = s_B \nu {s_A}^{-1} : \varphi(A) \mapsto \varphi(B)$.

Definition 2. *An automorphism $\varphi : C \mapsto C$ of category is called inner if*

1. φ is a hereditary automorphism.

2. For the substitution φ there exists a function s such that $\varphi = \hat{s}$. The equality $\hat{s}(\nu) = \varphi(\nu) = s_B \nu {s_A}^{-1}$ can be written as a commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{\nu} & B \\ {\scriptstyle s_A}\downarrow & & \downarrow{\scriptstyle s_B} \\ \varphi(A) & \xrightarrow{\varphi(\nu)} & \varphi(B). \end{array}$$

The diagram means the natural transformation of functors $s : 1_C \mapsto \varphi$ is an isomorphism of functors. Thus, an automorphism $\varphi : C \mapsto C$ is inner, if and only if φ is isomorphic to the identity automorphism $1_C : C \mapsto C$. Note that two automorphisms $\varphi_1, \varphi_2 : C \mapsto C$ are isomorphic if and only if ${\varphi_1}^{-1}\varphi_2$ is inner.

Definition 3. *A variety Θ is called perfect if the category of free algebras Θ^0 is perfect, that is, if every automorphism $\varphi : \Theta^0 \mapsto \Theta^0$ is inner.*

We study the question of perfectness of the variety $\mathfrak{B}_n$ of all n - periodic groups with odd exponent $n \geq 1003$. In this category the category ${\mathfrak{B}_n}^0$ of all free in $\mathfrak{B}_n$ algebras is exactly the category of free periodic (free Burnside) groups $B(m, n) = B(|X|, n)$, where X is a finite group alphabet ($|X| = m$).
For this category we obtained the following result.

Theorem. *The variety of n-periodic groups is perfect for any odd period* $n \geq 1003$.
It should be noted that the perfectness of the variety of all groups was proved in [3].

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Duality and bounded operators in spaces of harmonic functions

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Introduction. The following notations are used:
$B = \{x \in \mathbb{R}^n : \ |x| < 1\}$ stands for the open unit ball in $\mathbb{R}^n$ and
$S = \{x \in \mathbb{R}^n : \ |x| = 1\}$ for its boundary, i.e. the unit sphere in $\mathbb{R}^n$;
$h(B)$ means the vector space of complex-valued functions which are harmonic in B with the usual pointwise addition and scalar multiplication;
σ stands for the Lebesgue measure of the area element on S normed by the condition $\sigma(S) = 1$.

A positive continuous decreasing function φ on $[0, 1)$ is called *weight function* if $\lim \varphi(r) = 0$, as $r \to 1$, and a positive finite Borel measure η on $[0, 1)$ is called *weighting measure* if it is not supported in any subinterval $[0, \rho)$, $0 < \rho < 1$. Let $h_\infty(\varphi)$ be the Banach space of functions $u \in h(B)$, with the norm $\|u\|_\varphi = \sup\{|u(x)|\varphi(|x|) : \ |x| < 1\}$ and let $h_0(\varphi)$ be the closed subspace of functions u with $|u(x)| = o\big(1/\varphi(|x|)\big)$ as $|x| \to 1$.

It has been shown by Rubel and Shields, [1] that $h_\infty(\varphi)$ is isometrically isomorphic to the second dual of $h_0(\varphi)$. It was posed and solved in [2] the duality problem of finding a weighting measure η such that

$$h^1(\eta) = \big\{v \in L^1(d\eta(r)\, d\sigma) : \ v \in h(B)\big\}$$

represents the intermediate space, the dual of $h_0(\varphi)$ and the predual of $h_\infty(\varphi)$, i.e. $h^1(\eta) \sim h_0(\varphi)^*$ and $h^1(\eta)^* \sim h_\infty(\eta)$. In this duality relations the pairing is given by

$$\langle u, v\rangle = \int_0^1 \int_S u(r\zeta)\overline{v(r\zeta)}\varphi(r)\, d\sigma(\zeta)\, d\eta(r), \quad u \in h_\infty(\varphi), \ \ v \in h^1(\eta). \quad (1)$$

In the indicated articles [1] and [2] the case $n = 2$ is considered. It is well known that in this case every harmonic function h has an expansion in a series on degrees z and $\overline{z}$ in the unit disk $|z| < 1$, since real-valued $h \in h(B)$ is a real part of a holomorphic function. It allows to apply the methods of complex analysis.

It is shown in [3] that the duality problem is solvable in spaces of function which are holomorphic in the unit ball of $\mathbb{C}^n$ if φ is normal. The essential part of the definition of "normal" is that $1/\varphi(r)$ grows slower than some power of $1/(1-r)$ but faster than some other power.

In the report, we consider duality problem for the case of harmonic functions in the unit ball of $\mathbb{R}^n$, $n > 2$. The multidimensional case has the specifics in the sense that we can not speak about connection between harmonic and holomorphic functions, and instead of degrees z and $\overline{z}$ we deal with spherical harmonics.
We use the same approach to the duality problem as in [4]. This approach is based on establishing a certain integral operator from $L^\infty(d\eta(r)\,d\sigma)$ to $h_\infty(\varphi)$ is a bounded projection. The kernel of the integral operator is the reproducing kernel (see [5] for details) associated with the bilinear form (1). It is supposed that weight function grows more slowly than some power of $1/(1-r)$. Thus, we have a solution of duality problem for non-normal weight functions as

$$\varphi(r) = \left(\ln \frac{e}{1-r}\right)^{-\alpha}, \quad \alpha > 0.$$

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Colored Petri Nets as a Context-free Languages Modeling System

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Introduction. Colored Petri Nets(CPN) is a generalization of the Classical PN, created by K. Jensen [1, 2, 3]. CPN is a graphical oriented language for design, specification, simulation and verification of systems [3]. Particularly, it is well-suited for systems that consist of a number of communicated and synchronized processes. Typical examples of applications are communication protocols, distributed systems, automated production systems, work flow analysis.

Definition. *CPN is a nine-tuple* $(\Sigma, P, T, A, N, C, G, E, I)$*, where:*

1. Σ is a finite set of non-empty types, also called color sets.
2. P is a finite set of places.
   In the associated CPN Tool they are depicted as ovals/circles.
3. T is a finite set of transitions.
   In the associated CPN Tool they are depicted as rectangles.
4. A is a finite set of arcs: $P \cap T = P \cap A = T \cap A = \emptyset$.
5. N is a node function, defined from A into $P \times T \cup T \times P$.
6. C is a color function, $C : P \to \Sigma$:
   $$t \in T : [Type(G(t)) = B\&Type(Var(G(t))) \subseteq \Sigma].$$
7. G is a guard function. It is defined from T into expressions as
   $$t \in T : [Type(G(t)) = B\&Type(Var(G(t))) \subseteq \Sigma].$$
8. E is an arc expression function:
   $$\forall a \in A : [Type(E(a)) = C(p)_{MS}\&Type(Var(E(a))) \subseteq \Sigma].$$
9. I is an initialization function: $\forall p \in P : [Type(I(p)) = C(p)_{MS}]$.

The distribution of tokens, called marking, in the places of a CPN determines the state of a system being modeled.

The dynamic behavior of a CPN is described in terms of the firing of transitions. The firing of a transition takes the system from one state to another. A transition is enabled if the associated arc expressions of all incoming arcs can be evaluated to a multi-set, compatible with the current tokens in their respective input places, and its guard is satisfied.

Unlike Regular languages there are Context-free languages, which are not languages of PN. As an example we note the following ([1, 4]):

$$\{\omega\omega^n / \omega \in \Sigma^*,\ \Sigma = \{a, b\}\} \quad .$$

This fact illustrates the limitation of PN as a language generated tool.

It is impossible in PN to remember arbitrarily long sequence of characters. However, in PN finite length sequences could be remembered (which is also possible in finite automata) [1]. However, PN have not the "capacity

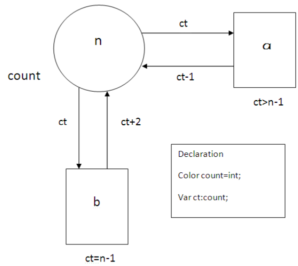

Figure 1: Modeling $\{\omega\omega^n/\omega \in \Sigma^*,\ \Sigma = \{a, b\}\}$ CF language by CPN.

of pushdown memory" which is necessary for the generation of Context-free (CF) languages. The interrelation of languages of PN with other classes of languages was studied by Ven [1].
The figure 1 shows a CPN, which generates the $\{\omega\omega^n/\omega \in \Sigma^*,\ \Sigma = \{a, b\}\}$ language which means that CPN is more powerful than the Classical PN. To understand types of data used in the Figure, it is necessary to give a declaration. One of the output ranges of $\{\omega\omega^n/\omega \in \Sigma^*,\ \Sigma = \{a, b\}\}$ language is the following: abaaba or babbab. Figure 1 models the first range. If transition-names are changed to each other, it will be generated the second one.
As a conclusion: CPN, due to its important properties, are more comfortable for system-modeling mentioned above.

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On the Construction of Point Processes in Statistical Mechanics

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We present a new approach to the construction of point processes of classical statistical mechanics as well as processes related to the Ginibre Bose gas of Brownian loops and to the dissolution in $\mathbb{R}^d$ of Ginibre's Fermi-Dirac gas of such loops. This approach is based on the cluster expansion method. We obtain the existence of Gibbs perturbations of a large class of point processes. Moreover, it is shown that certain "limiting Gibbs processes" are Gibbs in the sense of Dobrushin, Lanford and Ruelle if the underlying potential is positive. Finally, Gibbs modifications of infinitely divisible point processes are shown to solve a new integration by parts formula if the underlying potential is positive.

We reconsider the problem of construction of interacting point processes which are of importance in statistical physics. They include Gibbs processes of classical statistical mechanics and processes which are associated to continuous quantum systems in the sense of Ginibre, the so-called gases of winding Brownian loops.
Earlier approaches can be found in the works of Kondratiev et al. in the case of Boltzmann statistics, and in the thesis of Kuna. But several questions are left open here. The method we use is a new version of cluster expansions which had been developed in [2, 4] and which is summarized in our main theorem.
By combining this method with the abstract cluster expansions from [1] we then construct limiting interacting processes in the context of statistical mechanics. As a first application we consider the interacting quantum Bose gas in Feynman-Kac representation. This yields a point process corresponding to the Bose gas of interacting winding loops.
One of the main assumptions of the next result is the positivity of the reference measure. This is not the case for the Fermi loop gas where appears a *signed* reference measure. Therefore assuming hypothetically the existence of a cluster process for the Fermi gas of polygonal loops, and dissolving clusters into its particles, one would obtain a Gibbs modification of a determinantal point process in Euclidean space. We are able to

construct this process by means of our method. Such processes in a more general setting are presented in the work. As examples we consider Gibbs modifications of the Poisson as well as determinantal and permanental processes.
We also show that under natural regularity conditions the limiting processes are Gibbs in the sense of Dobrushin/Lanford/Ruelle (DLR) if the underlying interaction is positive. Finally, it is shown that Gibbs modifications with positive pair potential of infinitely divisible point processes solve a new integral equation involving the Campbell measure of the process. This equation generalizes the integration by parts formula of X.X. Nguyen et al. [3] which is equivalent to the DLR-equation. Examples of such processes are Gibbs modifications of the ideal Bose gas.

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On a problem for two-dimensional Dirac's system

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We consider the following boundary problem for two-dimensional Dirac's system [1], [2]:

$$\ell\overline{y} = \lambda\overline{y}, \qquad \overline{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \tag{1}$$

$$y_4(0) = h_1 y_1(0), \tag{2}$$

$$y_3(0) = h_2 y_2(0), \tag{3}$$

where

$$\ell = S\frac{d}{dx} + \Omega(x), \quad S = \frac{1}{i}\sigma_1, \quad \Omega(x) = \sigma_2 p(x) + \sigma_3 q(x) + \sigma_4 r(x),$$

$\sigma_i \ \ (i = 1, 2, 3, 4)$ being Dirac matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$p, q, r \in C[0, \infty)$, $\ h_i \ (i = 1,\ 2)$ are arbitrary complex numbers, λ is a parameter.

Let

$$A_1 = S\frac{d}{dx} + \Omega(x), \quad A_2 = S\frac{d}{dx}, \quad 0 \le 0 < \infty,$$

and let E denote the space of four-component vector-functions

$$\overline{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{pmatrix}, \ \ 0 \le x < \infty$$

which are continuous and have continuous first derivatives, E_1 and E_2 are subspaces of E consisting of vector-functions, satisfying the boundary condition

$$f_4(0) = h_1 f_1(0), \quad f_3(0) = h_2 f_2(0).$$

Definition. *Linear invertible operator (4×4 matrix) X, defined on the whole E and acting from E_1 to E_2 is called transformation operator for a pair of operators A_1 and A_2, if the following conditions are satisfied:*

1. *Operator X and its inverse X^{-1} are continuous on the space E.*

2. *The identity $A_1 X = X A_2$ holds.*

Theorem. *Operator X, transforming E_1 onto E_2, is implemented as*

$$X\overline{f}(x) = \overline{f}(x) + \int_0^x L(x,t)\overline{f}(x)dt. \tag{4}$$

Kernel-matrix $L(x,t)$ of the operator (4) is a solution of the differential equation

$$SL'_x(x,t) + L'_t(x,t)S = -\Omega(x)L(x,t) \tag{5}$$

and satisfies the conditions

$$L(x,x)S - SL(x,x) = \Omega(x), \tag{6}$$

$$L(x,0)SH = 0, \quad H = \begin{pmatrix} 1 \\ 1 \\ h_2 \\ h_1 \end{pmatrix}. \tag{7}$$

Conversely, if kernel-matrix $L(x,t)$ is a solution of the problem (5)-(7), then operator X, defined by the formula (4), is a transformation operator for the pair of operators A_1 and A_2, acting from E_1 to E_2.

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Some quadrature and cubature formulas of Hermite type

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In 1814 C. F. Gauss published his famous quadrature formula of the highest algebraic degree of accuracy. The Gaussian formula opened the central direction in numerical analysis devoted to the construction of approximation methods of a given form that are exact for all polynomials of the highest possible degree. The Gaussian setting reflected the classical understanding that regards the algebraic polynomials as simple and nice functions which can easily be differentiated, integrated, and evaluated at any point. Moreover, by the Weierstrass theorem, an arbitrary continuous function can be approximated by polynomials with arbitrary accuracy. For this reason, every approximation method which is good for a broad class of polynomials was regarded as a good method. Starting from the very appearance of the first computers, one can see an active consideration of theoretical problems related to the solution of extremal problems on classes of functions.
At that time (in the 1940s) A. Kolmogorov posed the following problem:
Construct a quadrature formula of a given type that has minimal error on a given class of functions.
However, the optimal quadrature formulas may not be unique. More information about this can be found in a review article of B. Boyanov, [1].
The main part of the report is devoted to the quadrature and cubature formulas of Hermitian type with the highest trigonometric degree of accuracy. The report provides some quadrature and cubature formulas of Hermitian type, obtained by the author. The convergence of the quadratures is studied, as well as error estimates. The numerical analysis of the formulas is held. The obtained formulas are compared with some well-known formulas.

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Nonself-adjoint degenerate ordinary differential equations of higher order

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The main object of the present talk is the ordinary differential equation

$$Su \equiv (-1)^m (t^\alpha u^{(m)})^{(m)} + a(t^{\alpha-1}u^{(m)})^{(m-1)} + pt^{\alpha-2m}u = f, \qquad (1)$$

where $m \in \mathbb{N}, t \in (0,b), \alpha \geq 0, \alpha \neq 1,3,\dots,2m-1$, a and p are real constants, $f \in L_{2,-\alpha}$, i.e., $\|f\|^2_{L_{2,-\alpha}} = \int_0^b t^{-\alpha}|f(t)|^2\,dt < \infty$.
Let $\dot{W}^m_\alpha$ be the completion of $\dot{C}^m := \{u \in C^m[0,b], u^{(k)}(0) = u^{(k)}(b) = 0$, $k = 0,1,\dots,m-1\}$ in the norm $\|u\|^2_{\dot{W}^m_\alpha} = \int_0^b t^\alpha |u^{(m)}(t)|^2\,dt$. It is known (see, for instance, [4]) that for $\beta \geq \alpha - 2m$ there is a continuous embedding $\dot{W}^m_\alpha \hookrightarrow L_{2,\beta}$, which is compact for $\beta > \alpha - 2m$.

Definition 1. *A function $u \in \dot{W}^m_\alpha$ is called generalized solution of the equation* (1) *if for every $v \in \dot{W}^m_\alpha$ the following equality holds*

$$(t^\alpha u^{(m)}, v^{(m)}) + a(-1)^{m-1}(t^{\alpha-1}u^{(m)}, v^{(m-1)}) + p(t^{\alpha-2m}u, v) = (f, v),$$

where $(\cdot,\cdot)$ stands for the scalar product in $L_2(0,b)$.

Let $d(m,\alpha) = 4^{-m}(\alpha-1)^2(\alpha-3)^2\cdots(\alpha-(2m-1))^2$. Observe, that we have exact inequality $\int_0^b t^{\alpha-2m}|u(t)|^2\,dt \leq d(m,\alpha)\int_0^b t^\alpha |u^{(m)}(t)|^2\,dt$, for $u \in \dot{W}^m_\alpha$.

Theorem 1. *Let the following condition be fulfilled*

$$a(\alpha-1)(-1)^m > 0, d(m,\alpha) + \frac{a}{2}(\alpha-1)(-1)^m d(m-1)(\alpha-2) + p > 0. \quad (2)$$

Then the equation (1) *has a unique generalized solution for every $f \in L_{2,-\alpha}$.*

Define an operator $S : L_{2,\alpha} \to L_{2,-\alpha}$ corresponding to the Definition 1 (see [2], [4]). Denote $\mathbb{S} := t^{-\alpha}S, D(\mathbb{S}) = D(S)$. Thus, we get an operator in the space $L_{2,\alpha}$.

Theorem 2. *Under the assumptions of the Theorem 1, the inverse operator $\mathbb{S}^{-1} : L_{2,\alpha} \to L_{2,\alpha}$ is compact.*

For the case $a(\alpha-1)(-1)^m>0$, let us consider the equation

$$Tv \equiv (-1)^m (t^\alpha v^{(m)})^{(m)} - a(t^{\alpha-1} v^{(m-1)})^{(m)} + p t^{\alpha-2m} v = g, g \in L_{2,-\alpha}. \quad (3)$$

Definition 2. *We say that* $v \in L_{2,\alpha}$ *is the generalized solution of* (3) *if for every* $u \in D(S)$ *the following equality holds*

$$(Su, v) = (u, g).$$

Let $g = t^\alpha g_1, g_1 \in L_{2,\alpha}$.
Then, as above, we obtain an operator $\mathbb{T} : L_{2,\alpha} \to L_{2,\alpha}, D(\mathbb{T}) = D(T)$. Note that it is adjoint to the operator $\mathbb{S}$, i.e., $\mathbb{T} = \mathbb{S}^*$.

Theorem 3. *If the condition* (2) *is fulfilled then a generalized solution of the equation* (3) *exists and is unique, for every* $g \in L_{2,-\alpha}$*. Moreover, the inverse operator* $\mathbb{T}^{-1} : L_{2,\alpha} \to L_{2,\alpha}$ *is compact.*

At last, we note that for $u \in D(S)$ we have only finiteness of the value $t^{\alpha-1}|u^{(m-1)}(t)|^2|_{t=0}$, while for $u \in D(T)$ this value is equal to zero. This is some analogue of the Keldysh effect (see [1], [3]).

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On spectrum of Sturm-Liouville operator

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Introduction. In the report the Sturm - Liouville operator on the finite interval is considered. For special boundary conditions a group of invariant transformations preserving the operator's spectrum is constructed. This result permits to find out conditions under which the spectrum is discrete. The influence of this group on the inverse problem is discussed. Similar results one can find in [1 - 7].

The following results are presented.

Theorem 1. Let the operator

$$-y''(x) = \lambda p(x)y(x), \quad -1 < x < 1,$$

with boundary conditions $y(-1) - ay'(-1) = 0$, $y(1) + by'(1) = 0$ have discrete spectrum. If $2a + a^2 = 0$ and $2b + b^2 = 0$, then for arbitrary $x_0 \in (-1, 1)$ the potential function

$$\frac{(1 - x_0^2)^2}{(1 - xx_0)^4}\, p\left(\frac{x - x_0}{1 - xx_0}\right)$$

generates an operator with the same spectrum as $p(x)$ does.
If the additional condition $a = b$ holds, the potential function $p(-x)$ generates an operator with the same spectrum.

Theorem 2. Let $p(x) \geq 0$ be a measurable function, $-1 \leq x \leq 1$.
Then the spectrum of the operator

$$-y''(x) = \lambda p(x)y(x), \quad -1 \leq x \leq 1,$$

with the boundary conditions $y(-1) = y(1) = 0$ is discrete in the space of functions for which $\int_{-1}^{1} |y'(x)|^2 dx < \infty$ if and only if

$$\sup_{|x|<1} \int_{2|t-x|<1-|x|} \sqrt{p(t)}dt < \infty$$

and $\lim_{|x|\to 1^-} \int_{2|t-x|<1-|x|} \sqrt{p(t)}dt = 0$.

Theorem 3. Let $p(x) \geq 0$ be a measurable function, $-1 \leq x \leq 1$.
Then the spectrum of the operator

$$-y''(x) = \lambda p(x)y(x), \quad -1 \leq x \leq 1,$$

with boundary conditions $y(-1) = y(1) = 0$ is discrete in the space of functions for which $\int_{-1}^{1} y^2(x)(1-x^2)^{-2}dx < \infty$ if and only if

$$\sup_{|x|<1} \left((1-|x|)^3 \int_{2|t-x|<1-|x|} p^2(t)dt \right) < \infty.$$

and

$$\lim_{|x|\to 1^-} \left((1-|x|)^3 \int_{2|t-x|<1-|x|} p^2(t)dt \right) = 0.$$

Theorem 4. Let the real numbers a, b satisfy the condition

$$(1-(a+1)^2)(1-(b+1)^2) > 0.$$

Let $x_0 = (a-b)/(a+b+ab) \in (-1,1)$, (we put $x_0 = 0$ if $a = b$). Let $p(x) > 0$ be a continuous differentiable function satisfying the condition

$$(1-x^2)^2 p(x) = (1-y^2)^2 p(y), \quad x, y \in (-1,1), \quad \frac{x+y}{1+xy} = x_0.$$

Let $\lambda_0 < \lambda_1 < \lambda_2 < \dots$ be the spectrum of the operator

$$-y''(x) = \lambda p(x)y(x), \quad y(-1) - ay'(-1) = 0, \quad y(1) + by'(1) = 0.$$

Then there is no other potential function $p(x)$ satisfying the conditions given above and generating the operator with the same spectrum.

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